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SPONTANEOUS DIMENSION REDUCTION AND THE EXISTENCE OF A LOCAL LAGRANGIAN FOR GIVEN n-DIMENSIONAL NEWTONIAN EQUATIONS OF MOTION

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ABSTRACT: A partially explicit construction of a Lagrangian for an n-dimensional Newtonian system of equations of motion is given. Extra variables used in the construction are spontaneously reduced by the constraints resulting from degeneracy of the proposed Lagrangian, so that only the variables that appear in the original system of equations remain. An explicit example of a Lagrangian for a system not satisfying Helmholtz conditions is given.

INDEX WORDS: Lagrangian, Newtonian system of equations.

I. Introduction

While the Newtonian equations of motion are physically more fundamental, the Lagrangian that would produce these equations and the Hamiltonian resulting from it are the most accepted way of describing mechanical systems. The problem of constructing a Lagrangian for given equations of motion was therefore extensively studied, but it is still not completely resolved.

It is well known that a Lagrangian for an arbitrary Newtonian system of equations of motion can be constructed by placing these equations directly into that Lagrangian and multiplying them by additional variables, as it is done in Bateman-Morse-Feshbach approach [1,2]. The method is somewhat analogous to Lagrange multipliers used for imposing holonomic constraints on a mechanical system. However, this approach creates additional non-physical variables that were not existing in the original equations of motion. These additional variables are then present in the Lagrange-Hamilton formalism, and it is not clear how to interpret them. So this approach, while relatively simple, is not commonly accepted as a resolution of the Lagrangian construction problem.

Another approach is to require (implicitly) that a Lagrangian will be restricted to only these variables that are already present in the Newtonian equations of motion. If we also require that the equations of motion are directly produced from that Lagrangian as its Euler-Lagrange equations, then some very basic physical systems would have no Lagrangian [3]. The most common resolution of this problem is to allow modification of the original equations of motion by so called integral multipliers. With this modification it turns out that Lagrangians always exist for one-dimensional Newtonian equation [4]. In dimensions two or higher only some Newtonian systems modified by integral multipliers allow a Lagrangian. A complete characterization of such systems was only done for two dimensional systems, and it turned out to be unexpectedly complicated [5]. There exist many studies for dimensions higher than two [6,7], but still there exists no way that would allow to look at an arbitrary system of Newtonian equations and decide if that system, with the use of integral multipliers, allows a Lagrangian that is using only the variables that appear in that Newtonian system.

In our approach to create a Lagrangian, we use variables that do not appear in the original equations of motion, as well as those that do. The Euler-Lagrange equations obtained from our Lagrangian reproduce the original equations of motion, as required. No integral multipliers are needed. More important, we avoid the basic difficulty of the Bateman-Morse-Feshbach approach, because some of the Euler-Lagrange equations obtained from our Lagrangian are constrains rather than differential equations. These constraints then turn out to eliminate all variables that do not appear in the original equations of motion, while leaving the original equations of motion intact.

In this work we are restricting ourselves to equations of motion and Lagrangians defined locally, and all statements below refer to the local situations only.
Our approach will in principle work for all Newtonian systems of equations, proving that all of them have a Lagrangian. In practice, obtaining that Lagrangian explicitly may be difficult or impossible, since it will require explicit calculation of the flow-box coordinates [8] for the given set of ordinary differential equations.

The organization of our presentation is as follows:

In section II, we discuss the "free particle" variables for a given set of Newtonian equations.

In section III, we present the proposed Lagrangian and we discuss its Euler-Lagrange equations of motion. We show that some of these equations are constraints and that these constraints eliminate all the additional variables used to create the Lagrangian, leaving only the variables that were present in the original Newtonian equations. The original variables left are still satisfying the original Newtonian equations of motion, as expected.

In section IV, we present a specific example of our method. By using Helmholtz conditions [6], we show that this example has no Lagrangian that would contain the coordinates from the equations of motion only, even with possible integrating multipliers. Yet, our method produces an explicit Lagrangian for this example.

II. The "free particle" coordinate system

We will start with an n-dimensional second order system of equations. Let us assume that the equations of motion are of the second order, and they can be solved for the second derivatives, so they are given in the form ($i = 1,\ldots,n$):

$$\ddot{x}_i = R_i(x, \dot{x}) \qquad (2.1)$$

where the coordinates in the n-dimensional configuration space are $x = (x_1, x_2, \ldots, x_n)$, and the dot above a variable denotes the time derivative. Often equations of that kind are called Newtonian.

Introducing velocity variables $v_i = \dot{x}_i$ we can replace equations (2.1) by first order equations ($i = 1,\ldots,n$):

$$\dot{x}_i = v_i$$
$$\dot{v}_i = R_i(x, v) \tag{2.2}$$

Arguably, the simplest physical system of the (2.2) kind is an n-dimensional free particle of unitary mass. Its equations of motion are ($i = 1,\dots,n$):

$$\dot{y}_i = w_i$$
$$\dot{w}_i = 0 \tag{2.3}$$

where $y = (y_1, y_2, \dots, y_n)$ are coordinates in n-dimensional configuration space, and $w = (w_1, w_2, \dots, w_n)$ are the velocities.

Mathematically, there exists even a simpler system, called the “flow box” or “straightened out” system, in which equations are

$$\dot{z}_1 = 1$$
$$\dot{z}_i = 0 \qquad i = 2,\dots,2n. \tag{2.4}$$

where $z = (z_1, z_2, \dots, z_{2n})$ are coordinates in a $2n$-dimensional coordinate space.

The free particle system (2.3) can be locally related to the general system (2.2) via the “flow box” system (2.4). Namely, the “flow box” theorem [8], when used for the systems like (2.2) states that for any point at which the right sides of the equations (2.2) are not all equal to 0, there exists a local invertible change of variables ($i = 1,\dots,n$)

$$z_i = z_i(x, v) \tag{2.5}$$

such that the equations (2.4), expressed by the variables (2.5), become equations (2.2), and vice-versa.

Similarly, there exists a local invertible change of variables ($i = 1,\dots,n$)

$$z_i = z_i(y, w) \tag{2.6}$$

such that the equations (2.4), expressed by the change of variables (2.6), become equations (2.3), and vice-versa. (We use the same symbol z_i for changes of variables in (2.5) and (2.6), since we do not intend to use them for long. The changes of variables are obviously not identical in (2.5) and 2.6)).

Because of the invertibility of the variable changes (2.5) and (2.6) we also have change of variables (made of composition of (2.5) and (2.6) and their inverses) ($i=1,\ldots,n$)

$$\begin{aligned} x_i &= f_i(y,w) \\ v_i &= g_i(y,w) \end{aligned} \tag{2.7}$$

and its inverse

$$\begin{aligned} y_i &= k_i(x,v) \\ w_i &= s_i(x,y). \end{aligned} \tag{2.8}$$

These changes of variables are such that equations (2.2), when expressed in variables (y,w), become (2.3). And equations (2.3), when expressed in variables (x,v), become (2.2). This means that the time derivatives of the equations (2.7), when expressed by the time derivatives of (y,w) and compared with the equations (2.2), give ($i=1,\ldots,n$):

$$\begin{aligned} \sum_{j=1}^{n} \frac{\partial f_i(y,w)}{\partial y_j}\cdot \dot{y}_j + \sum_{j=1}^{n} \frac{\partial f_i(y,w)}{\partial w_j}\cdot \dot{w}_j &= v_i \\ \sum_{j=1}^{n} \frac{\partial g_i(y,w)}{\partial y_j}\cdot \dot{y}_j + \sum_{j=1}^{n} \frac{\partial g_i(y,w)}{\partial w_j}\cdot \dot{w}_j &= R_i(x,v) \end{aligned} \tag{2.9}$$

Now using (2.2) we get ($i=1,\ldots,n$)

$$\begin{aligned} \sum_{j=1}^{n} \frac{\partial f_i(y,w)}{\partial y_j}\cdot w_j &= v_i \\ \sum_{j=1}^{n} \frac{\partial g_i(y,w)}{\partial y_j}\cdot w_j &= R_i(x,v) \end{aligned} \tag{2.10}$$

Similarly the time derivatives of the equations (2.8), when expressed by the time derivatives of (x,v) and compared with the equations (2.3), give ($i=1,\ldots,n$)

$$\sum_{j=1}^{n} \frac{\partial k_i(x,v)}{\partial x_j} \cdot \dot{x}_j + \sum_{j=1}^{n} \frac{\partial k_i(x,v)}{\partial v_j} \cdot \dot{v}_j = w_i$$
$$\sum_{j=1}^{n} \frac{\partial s_i(x,v)}{\partial x_j} \cdot \dot{x}_j + \sum_{j=1}^{n} \frac{\partial s_i(x,v)}{\partial v_j} \cdot \dot{v}_j = 0 \qquad (2.11)$$

Now using (2.3) we get ($i = 1,\ldots,n$)

$$\sum_{j=1}^{n} \frac{\partial k_i(x,v)}{\partial x_j} \cdot v_j + \sum_{j=1}^{n} \frac{\partial k_i(x,v)}{\partial v_j} \cdot R_j(x,v) = w_i$$
$$\sum_{j=1}^{n} \frac{\partial s_i(x,v)}{\partial x_j} \cdot v_j + \sum_{j=1}^{n} \frac{\partial s_i(x,v)}{\partial v_j} \cdot R_j(x,v) = 0 \qquad (2.12)$$

III. The Lagrangian

Consider the Lagrangian given by

$$L = \sum_{j=1}^{n} \frac{w_j^{\,2}}{2} + \sum_{j=1}^{n} \lambda_j (\dot{y}_j - w_j) + \sum_{j=1}^{n} \mu_j (x_j - f_j(y,w)) + \sum_{j=1}^{n} \eta_j (v_j - g_j(y,w))$$

(3.1)

where the functions f_j and g_j are such as defined in (2.7) and variables λ_j, μ_j and η_j are new variables, treated on equal footing with all other variables.

The Euler-Lagrange equations are then ($j = 1,\ldots,n$):

(associated with w_j)

$$w_j = \lambda_j + \sum_{i=1}^{n} \mu_i \cdot \frac{\partial f_i(y,w)}{\partial w_j} + \sum_{i=1}^{n} \eta_i \frac{\partial g_i(y,w)}{\partial w_j} \qquad (3.2i)$$

(associated with y_j)

$$\dot{\lambda}_j = -\sum_{i=1}^{n} \mu_i \cdot \frac{\partial f_i(y,w)}{\partial y_j} - \sum_{i=1}^{n} \eta_i \frac{\partial g_i(y,w)}{\partial y_j} \qquad (3.2ii)$$

(associated with x_j)

$$\mu_j = 0 \qquad (3.2iii)$$

(associated with v_j)

$$\eta_j = 0 \qquad (3.2iv)$$

(associated with μ_j)

$$x_j = f_j(y, w) \qquad (3.2v)$$

(associated with η_j)

$$v_j = g_j(y, w) \qquad (3.2vi)$$

(associated with λ_j)

$$\dot{y}_i = w_i \qquad (3.2vii)$$

Substituting (3.2iii) and (3.2iv) into (3.2i) and (3.2ii) we get ($j = 1, \ldots, n$)

$$w_j = \lambda_j \qquad (3.3i)$$

$$\dot{\lambda}_j = 0 \qquad (3.3ii)$$

$$\mu_j = 0 \qquad (3.3iii)$$

$$\eta_j = 0 \qquad (3.3iv)$$

$$x_j = f_j(y, w) \qquad (3.3v)$$

$$v_j = g_j(y, w) \qquad (3.3vi)$$

$$\dot{y}_i = w_i \quad (3.3vii)$$

The equations (3.3i), (3.3iii), (3.3iv), (3.3v), and (3.3vi) are not differential equations, therefore they represent constrains. We use these constraints to simplify other equations. Also, the time derivatives of the constraints must be zero, because they hold over time. Therefore time derivatives of constraints may produce equations giving time derivatives of some variables or new constraints. Time derivatives of the new constraints must again be equal to zero, which may produce more time derivatives and more constraints. We continue the process until no new constraints and no new time derivatives are produced. The result of this rather tedious calculation is ($i = 1,\ldots,n$):

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= R_i(x,v) \end{aligned} \quad (3.4)$$

$$\begin{aligned} \lambda_i &= s_i(x,y) \\ y_i &= k_i(x,v) \\ w_i &= s_i(x,y) \end{aligned} \quad (3.5)$$

$$\begin{aligned} \dot{\lambda}_i &= \sum_{j=1}^{n} \frac{\partial s_i(x,v)}{\partial x_j} \cdot v_j + \sum_{j=1}^{n} \frac{\partial s_i(x,v)}{\partial v_j} \cdot R_j(x,v) \\ \dot{y}_i &= \sum_{j=1}^{n} \frac{\partial k_i(x,v)}{\partial x_j} \cdot v_j + \sum_{j=1}^{n} \frac{\partial k_i(x,v)}{\partial v_j} \cdot R_j(x,v) \\ \dot{w}_i &= \sum_{j=1}^{n} \frac{\partial s_i(x,v)}{\partial x_j} \cdot v_j + \sum_{j=1}^{n} \frac{\partial s_i(x,v)}{\partial v_j} \cdot R_j(x,v) \end{aligned} \quad (3.6)$$

$$\begin{aligned} \mu_i &= 0 \\ \eta_i &= 0 \end{aligned} \quad (3.7)$$

$$\dot{\mu}_i = 0$$
$$\dot{\eta}_i = 0 \qquad (3.8)$$

When we look at the Euler-Lagrange equations (3.4)-(3.8) above, we can conclude that the equations (3.4) are the actual equations that give us the dynamics of the system. They are identical to the Newtonian equations (2.2) that we wanted to obtain from our Lagrangian.

The equations (3.5) and (3.7) are merely the definitions of variables $(\lambda, y, w, \mu, \eta)$ in terms of the variables (x, v). The equations (3.6) and (3.8) are merely the time derivatives of the equations (3.5) and (3.7). Therefore the variables $(\lambda, y, w, \mu, \eta)$ do not have any dynamics of their own; their dynamics is fully described by the dynamics of the variables (x, v). Recall that when we started with the Lagrangian (3.1), all the variables $(x, v, y, w, \lambda, \mu, \eta)$ were treated by us on equal footing, as independent variables. Yet, the Euler-Lagrange equations for that Lagrangian, by itself, reduced the role of the variables $(y, w, \lambda, \mu, \eta)$ to either some non-dynamical functions of (x, v) or some constants, leaving (x, v) as the only "real" variables, with "real" dynamics.

The situation above can be best described as a "spontaneous reduction of dimension."

To better understand this situation we may notice that the final result is not different from, let's say, having a two-dimensional free particle system described by variables (x_1, v_1, x_2, v_2) and satisfying the equations:

$$\dot{x}_1 = v_1 \qquad (3.9i)$$

$$\dot{v}_1 = 0 \qquad (3.9ii)$$

$$\dot{x}_2 = v_2 \qquad (3.9iii)$$

$$\dot{v}_2 = 0 \qquad (3.9iv)$$

We can then define another variable, for example

$$r = \sqrt{x_1^2 + x_2^2}. \qquad (3.10)$$

Then if the new variable and its time derivative calculated using equations (3.9) were incorporated into the system of equations (3.9), it would not change the fundamental fact that the resulting system would still describe two dimensional free particle system. The extra variable defined by (3.10) would not change anything.

Concluding, the Lagrangian (3.1) reproduces the original Newtonian equations of motion (2.2) as its Euler-Lagrange equations, while all additional variables in it are removed by other Euler-Lagrange equations of the same Lagrangian. No integral multipliers were needed.

IV. An Example

A traditional approach (the approach that uses only the variables that are present in the original equations of motion) to the existence of a Lagrangian for a given system of Newtonian equations of the type ($j = 1,\ldots,n$)

$$\ddot{x}_j(t) = F_j\left(x, \dot{x}, t\right) \qquad (4.1)$$

begins with introducing of so called “integrating multipliers.” They are needed because the requirement that the equations (4.1) are to be obtained directly as the Euler-Lagrange equations of a Lagrangian is too rigid. It would eliminate some physically basic systems, for example, one-dimensional particle under the influence of the frictional force which is linear in velocity [3]. Therefore the existence of a set of functions $\mu_{ij}\left(x, \dot{x}, t\right), \quad \det\left[\mu_{ij}\right] \neq 0$ called “integral multipliers” is postulated. The integral multipliers are such that the equations

$$\sum_j \mu_{ij}\left(x, \dot{x}, t\right) \cdot \left[F_j\left(x, \dot{x}, t\right) - \ddot{x}_j(t)\right] = 0 \qquad (4.2)$$

give a better chance to be obtained as Euler-Lagrange equations of some Lagrangian. The conditions for such multipliers were first studied by Helmholtz

[6]. A version of Helmholtz conditions convenient for our purposes was given by Dodonov et all.[3] as:

$$\mu_{ij} = \mu_{ji} \tag{4.3i}$$

$$\partial\mu_{ij} / \partial \dot{x}_k = \partial\mu_{jk} / \partial \dot{x}_i = \partial\mu_{ki} / \partial \dot{x}_j \tag{4.3ii}$$

$$\frac{d}{dt}\mu_{ij} + \sum_k \left[\frac{\mu_{ik}}{2} \cdot \partial F_k / \partial \dot{x}_j + \cdot \partial F_k / \partial \dot{x}_i \cdot \frac{\mu_{kj}}{2} \right] = 0 \tag{4.3iii}$$

$$\frac{d}{dt}\left[\sum_k \left(\mu_{ik} \cdot \partial F_k / \partial \dot{x}_j - \partial F_k / \partial \dot{x}_i \cdot \mu_{kj} \right) \right] + \\ -2\left[\sum_k \left(\mu_{ik} \cdot \partial F_k / \partial x_j - \partial F_k / \partial x_i \cdot \mu_{kj} \right) \right] = 0 \tag{4.3iv}$$

where

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_k \left(\dot{x}_k \cdot \frac{d}{dx_k} + F_k(x, \dot{x}, t) \cdot \frac{\partial}{\partial \dot{x}_k} \right) \tag{4.3v}$$

As our example let us consider the following system of equations:

$$\ddot{x}_1 = x_1 \tag{4.4i}$$

$$\ddot{x}_2 = \dot{x}_1 . \tag{4.4ii}$$

Thus, comparing with (4.1), we have

$$F_1 = x_1 \tag{4.5i}$$

$$F_2 = \dot{x}_1 \tag{4.5ii}$$

Using $i = j = 2$ in the equation (4.3iii) gives $\frac{d}{dt}\mu_{22} = 0$. Using $i = 2$ and $j = 1$ in (4.3iv) gives $\frac{d}{dt}\mu_{22} = 2\mu_{21}$. Consequently we have $\mu_{21} = 0$. Because of (4.3i), we also have and $\mu_{12} = 0$. Consequently $\frac{d}{dt}\mu_{12} = 0$. Using this and $i = 2$ and $j = 1$ in (4.3iii), gives $\frac{1}{2}\mu_{22} = 0$, so $\mu_{22} = 0$. Concluding, all entries in the matrix μ_{ij} are equal zero, so the condition $\det\lfloor\mu_{ij}\rfloor \neq 0$ cannot be satisfied. Therefore no traditional Lagrangian, meaning Lagrangian using only the original variables from the equations of motion, exists for the equations (4.4).

Introducing velocities, the equations (4.4) can be rewritten as

$$\dot{x}_1 = v_1 \tag{4.6i}$$

$$\dot{v}_1 = x_1 \tag{4.6ii}$$

$$\dot{x}_2 = v_2 \tag{4.6iii}$$

$$\dot{v}_2 = v_1 \tag{4.6iv}$$

Let us introduce new variables (y_1, w_1, y_2, w_2) by the formulas:

$$x_1 = w_1 \sinh\left(\frac{y_1}{w_1}\right) \tag{4.7i}$$

$$v_1 = w_1 \cosh\left(\frac{y_1}{w_1}\right) \tag{4.7ii}$$

$$x_2 = y_2 + w_1 \cosh\left(\frac{y_1}{w_1}\right) \tag{4.7i}$$

$$v_2 = w_2 + w_1 \sinh\left(\frac{y_1}{w_1}\right) \tag{4.7i}$$

A direct check shows that if we use the differential equations

$$\dot{y}_1 = w_1 \tag{4.8i}$$

$$\dot{w}_1 = 0 \tag{4.8ii}$$

$$\dot{y}_2 = w_2 \tag{4.8iii}$$

$$\dot{w}_2 = 0 \tag{4.8iv}$$

to calculate the time derivatives of equations (4.7), we will reproduce the equations (4.6).

We can invert the equations (4.7), getting

$$y_1 = \sqrt{{v_1}^2 - {x_1}^2}\ \tanh^{-1}\left(\frac{x_1}{v_1}\right) \tag{4.9i}$$

$$w_1 = \sqrt{{v_1}^2 - {x_1}^2} \tag{4.9ii}$$

$$y_2 = x_2 - v_1 \tag{4.9iii}$$

$$w_2 = v_2 - x_1 \tag{4.9iv}$$

A direct check shows that if we take time derivatives of equations (4.9) and then use equations of motion (4.6), we will reproduce the equations (4.8).

So, the variables (y_1, w_1, y_2, w_2) are the free particle variables for the system of equations (4.6), as defined in section II.

Following the general formula (3.1), we define the Lagrangian as:

$$L=\frac{w_1^{\ 2}}{2}+\frac{w_2^{\ 2}}{2}+\lambda_1(\dot{y}_1-w_1)+\lambda_2(\dot{y}_2-w_2)+$$
$$+\mu_1(x_1-w_1\sinh\left(\frac{y_1}{w_1}\right))+\mu_2(x_2-y_2-w_1\cosh\left(\frac{y_1}{w_1}\right))+$$
$$+\eta_1(v_1-w_1\cosh\left(\frac{y_1}{w_1}\right))+\eta_2(v_2-w_2-w_1\sinh\left(\frac{y_1}{w_1}\right))$$

(4.10)

All variables that appear n the Lagrangian above, namely $(x_1,x_2,v_1,v_2,y_1,y_2,w_1,w_2,\lambda_1,\lambda_2,\mu_1,\mu_2,\eta_1,\eta_2)$, are treated on the same footing, as independent. The Euler-Lagrange equations for that Lagrangian are then:

$$w_1=\lambda_1+\mu_1\sinh\left(\frac{y_1}{w_1}\right)-\mu_1\left(\frac{y_1}{w_1}\right)\cdot\cosh\left(\frac{y_1}{w_1}\right)+$$
$$+\mu_2\cosh\left(\frac{y_1}{w_1}\right)-\mu_2\left(\frac{y_1}{w_1}\right)\cdot\sinh\left(\frac{y_1}{w_1}\right)+$$
$$+\eta_1\cosh\left(\frac{y_1}{w_1}\right)-\eta_1\left(\frac{y_1}{w_1}\right)\cdot\sinh\left(\frac{y_1}{w_1}\right)+$$
$$+\eta_2\sinh\left(\frac{y_1}{w_1}\right)-\eta_2\left(\frac{y_1}{w_1}\right)\cdot\cosh\left(\frac{y_1}{w_1}\right)$$

(4.11i)

$$w_2=\lambda_2+\eta_2$$ (4.11ii)

$$\dot{\lambda}_1=-\mu_1\cosh\left(\frac{y_1}{w_1}\right)-\mu_2\sinh\left(\frac{y_1}{w_1}\right)$$
$$-\eta_1\sinh\left(\frac{y_1}{w_1}\right)-\eta_2\cosh\left(\frac{y_1}{w_1}\right)$$

(4.11iii)

$$\dot{\lambda}_2 = -\mu_2 \tag{4.11iv}$$

$$\mu_1 = 0 \tag{4.11v}$$

$$\mu_2 = 0 \tag{4.11vi}$$

$$\eta_1 = 0 \tag{4.11vii}$$

$$\eta_2 = 0 \tag{4.11viii}$$

$$x_1 = w_1 \sinh\left(\frac{y_1}{w_1}\right) \tag{4.11ix}$$

$$x_2 = y_2 + w_1 \cosh\left(\frac{y_1}{w_1}\right) \tag{4.11x}$$

$$v_1 = w_1 \cosh\left(\frac{y_1}{w_1}\right) \tag{4.11xi}$$

$$v_2 = w_2 + w_1 \sinh\left(\frac{y_1}{w_1}\right) \tag{4.11xii}$$

$$\dot{y}_1 = w_1 \tag{4.11xiii}$$

$$\dot{y}_2 = w_2 \tag{4.11xiv}$$

The equations (4.11i), (4.11ii), (4.11v), (4.11vi), (4.11vii), (4.11viii), (4.11ix), (4.11x), (4.11xi), and (4.11xii) are not differential equations, so they are constraints. We use these constraints to simplify other equations. Also, the time derivatives of the constraints must be zero, because they hold over time. Therefore time derivatives of constraints may produce equations giving time derivatives of some variables, or new constraints. Time derivatives of these new constraints must again be equal to zero, which may produce more time derivatives and more constraints. We continue the process until no new constraints and no new time derivatives are produced. The result of this somewhat tedious calculation is:

$$\lambda_1 = w_1 \qquad (4.12i)$$

$$\lambda_2 = w_2 \qquad (4.12ii)$$

$$\dot{\lambda}_1 = 0 \qquad (4.12iii)$$

$$\dot{\lambda}_2 = 0 \qquad (4.12iv)$$

$$\mu_1 = 0 \qquad (4.12v)$$

$$\mu_2 = 0 \qquad (4.12vi)$$

$$\eta_1 = 0 \qquad (4.12vii)$$

$$\eta_2 = 0 \qquad (4.12viii)$$

$$x_1 = w_1 \sinh\left(\frac{y_1}{w_1}\right) \qquad (4.12ix)$$

$$x_2 = y_2 + w_1 \cosh\left(\frac{y_1}{w_1}\right) \qquad (4.12x)$$

$$v_1 = w_1 \cosh\left(\frac{y_1}{w_1}\right) \qquad (4.12xi)$$

$$v_2 = w_2 + w_1 \sinh\left(\frac{y_1}{w_1}\right) \qquad (4.12xii)$$

$$\dot{y}_1 = w_1 \qquad (4.12xiii)$$

$$\dot{y}_2 = w_2 \qquad (4.12xiv)$$

$$\dot{w}_1 = 0 \qquad (4.12xv)$$

$$\dot{w}_2 = 0 \qquad (4.12xvi)$$

$$\dot{\mu}_1 = 0 \qquad (4.12xvii)$$

$$\dot{\mu}_2 = 0 \qquad (4.12xviii)$$

$$\dot{\eta}_1 = 0 \qquad (4.12xix)$$

$$\dot{\eta}_2 = 0 \qquad (4.12xx)$$

$$\dot{x}_1 = v_1 \qquad (4.12xxi)$$

$$\dot{x}_2 = v_2 \qquad (4.12xxii)$$

$$\dot{v}_1 = x_1 \qquad (4.12xxiii)$$

$$\dot{v}_2 = v_1 \qquad (4.12bxxiv)$$

Then we replace the equations (4.12ix), (4.12x), (4.12xi), and (4.12xii) with their inverses (4.9). We also replace variables (y_1, y_2, w_1, w_2) on the right side of all equations by (x_1, x_2, v_1, v_2) using equations (4.9). Finally, we change the order of equations getting:

$$\dot{x}_1 = v_1 \qquad (4.13i)$$

$$\dot{v}_1 = x_1 \qquad (4.13ii)$$

$$\dot{x}_2 = v_2 \qquad (4.13iii)$$

$$\dot{v}_2 = v_1 \qquad (4.13iv)$$

$$y_1 = \sqrt{{v_1}^2 - {x_1}^2} \ \tanh^{-1}\left(\frac{x_1}{v_1}\right) \qquad (4.14i)$$

$$w_1 = \sqrt{{v_1}^2 - {x_1}^2} \qquad (4.14ii)$$

$$y_2 = x_2 - v_1 \qquad (4.14iii)$$

$$w_2 = v_2 - x_1 \qquad (4.14iv)$$

$$\lambda_1 = \sqrt{{v_1}^2 - {x_1}^2} \qquad (4.14v)$$

$$\lambda_2 = v_2 - x_1 \qquad (4.14vi)$$

$$\mu_1 = 0 \qquad (4.14vii)$$

$$\mu_2 = 0 \qquad (4.14viii)$$

$$\eta_1 = 0 \qquad (4.14ix)$$

$$\eta_2 = 0 \qquad (4.14x)$$

$$\dot{y}_1 = \sqrt{{v_1}^2 - {x_1}^2} \qquad (4.15i)$$

$$\dot{w}_1 = 0 \qquad (4.15ii)$$

$$\dot{y}_2 = v_2 - x_1 \qquad (4.15iii)$$

$$\dot{w}_2 = 0 \qquad (4.15iv)$$

$$\dot{\lambda}_1 = 0 \quad (4.15v)$$

$$\dot{\lambda}_2 = 0 \quad (4.15vi)$$

$$\dot{\mu}_1 = 0 \quad (4.15vii)$$

$$\dot{\mu}_2 = 0 \quad (4.15viii)$$

$$\dot{\eta}_1 = 0 \quad (4.15ix)$$

$$\dot{\eta}_2 = 0 \quad (4.15x)$$

The equations (4.13), (4.14), and (4.15) are completely equivalent to the Euler-Lagrange equations (4.11) of the Lagrangian (4.10).

Equations (4.20) are identical to the original Newtonian equations of motion (4.6) which we wanted to obtain from our Lagrangian. They contain only the original variables (x_1, v_1, x_2, v_2), as required.

Equations (4.14) are merely definitions of all the additional variables $(y_1, w_1, y_2, w_2, \lambda_1, \lambda_2, \mu_1, \mu_2, \eta_1, \eta_2)$ used in the process, in terms of the original variables (x_1, v_1, x_2, v_2). Equations (4.15) are merely the time derivatives of the variables $(y_1, w_1, y_2, w_2, \lambda_1, \lambda_2, \mu_1, \mu_2, \eta_1, \eta_2)$.

Direct calculation shows that taking time derivatives of the definitions (4.14) using equations (4.13) reproduces the equations (4.15), as expected.

Therefore, as observed in section III, the existence of equations (4.14) and their time derivatives (4.15) is not changing the fact that the system given by the Lagrangian (4.10) is described by only the variables (x_1, v_1, x_2, v_2) and the Newtonian equations (4.6).

So, in this case we conclude that all the variables appearing in the Lagrangian (4.10) were spontaneously reduced to variables (x_1, v_1, x_2, v_2), which all appear in the original Newtonian equations of motion (4.4).

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