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# Spontaneous Dimension Reduction and the Existence of a local Lagrange-Hamilton Formalism for Given n-Dimensional Newtonian Equations of Motion

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# SPONTANEOUS DIMENSION REDUCTION AND THE EXISTENCE OF A LOCAL LAGRANGE-HAMILTON FORMALISM FOR GIVEN $n$-DIMENSIONAL NEWTONIAN EQUATIONS OF MOTION

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ABSTRACT: A partially explicit construction of a Lagrange-Hamiltonian formalism for an arbitrary $n$-dimensional Newtonian system of equations of motion is given. Additional variables used in the construction are spontaneously reduced by the Dirac's constraints resulting from degeneracy of the proposed Lagrangian, so that only the variables that appear in the original system of equations remain. A Hamiltonian and dynamical Dirac's brackets are calculated.

INDEX WORDS: Lagrangian, Hamiltonian, Newtonian systems, Poisson Bracket, Dirac's Bracket.

## I. Introduction

While the Newtonian equations of motion are physically more fundamental, the Lagrangian that would produce these equations and the Hamiltonian resulting from it are the most accepted way of describing mechanical systems. The problem of constructing a Lagrangian-Hamilton formalism for given equations of motion has been therefore extensively studied.

It is well known that a Lagrangian for an arbitrary Newtonian system of equations of motion can be constructed by placing these equations directly into that Lagrangian and multiplying them by additional variables, as it is done in Bateman-Morse-Feshbach approach [1,2]. The method is somewhat analogous to Lagrange multipliers used for imposing holonomic constraints on a mechanical system. However, this approach creates additional non-physical variables that were not existing in the original equations of motion. These additional variables are then present in the Lagrange-Hamilton formalism, and it is not clear how to interpret them. So this approach, while relatively simple, is not commonly accepted as a resolution of the Lagrangian construction problem.

Another approach is to require (implicitly) that a Lagrangian will be restricted to only these variables that are already present in the Newtonian equations of motion. If we also require that the equations of motion are directly produced from that Lagrangian as its Euler-Lagrange equations, then some very basic physical systems would have no Lagrangian [3]. The most common resolution of this problem is to allow modification of the original equations of motion by, so called, integral multipliers. With this modification it turns out that Lagrangians always exist for one-dimensional Newtonian equation [4]. In dimensions two or higher only some Newtonian systems modified by integral multipliers allow a Lagrangian. A complete characterization of such systems was only done for two dimensional systems, and it turned out to be unexpectedly complicated [5]. There exist many studies for dimensions higher than two [6,7], but still there exists no way that would allow to look at an arbitrary system of Newtonian equations and decide if that system, with the use of integral multipliers, allows a Lagrangian that is using only the variables that appear in that Newtonian system.

In our approach to create a Lagrangian [8], we used variables that did not appear in the original equations of motion, as well as those that did. The Euler-Lagrange equations obtained from our Lagrangian reproduced the original equations of motion, as required. Also, we avoided the basic difficulty of the Bateman-Morse-Feshbach approach, because some of the Euler-Lagrange equations obtained from our Lagrangian were constrains rather than differential equations. These constraints then turned out to eliminate all variables that did not appear in the original equations of motion, while leaving the original equations of motion intact.

In our work we are restricting ourselves to local description only. The equations of motion, Lagrangians and Hamiltonians, Poisson and Dirac's brackets, are all defined locally, and all statements below refer to the local situations only.

Our approach will in principle work for all Newtonian systems of equations, proving that all of them have a Lagrangian and Hamiltonian structure. In practice, obtaining that structure explicitly may be difficult or impossible, since it would require explicit calculation of the flow-box coordinates [9] for the given set of ordinary differential equations. Still, as we have shown in our previous work [8], at least some systems that do not have a traditional Lagrangian because they do not satisfy Helmholtz conditions [3,6], will nevertheless allow explicit construction of a Lagrangian using our method. In this work we will also show that once a Lagrangian is explicitly known, then the Hamiltonian and dynamical brackets can be calculated explicitly as well.

The organization of our presentation is as follows:

In section II, we start with recalling a Lagrangian for an arbitrary system of equations of motion, as given in our previous work [8]. We then present the Hamilton equations of motion for that Lagrangian. Some of these equations turn out to be constraints. We show that these constraints eliminate all the additional variables used to create the Lagrangian, leaving only the variables that were present in the original Newtonian equations. These original variables are still satisfying the original Newtonian equations of motion, as expected.

In section III, we explicitly calculate the Dirac's brackets [10,11] using all the constraints obtained in section II. The results for the Dirac's brackets of all variables are then expressed by using only the variables that appear in the original equations of motion. This makes the Dirac's brackets for all the original variables self-contained; the use of other variables is not needed anymore.

In section IV, we calculate the Hamiltonian and simplify it using the constraints from section II. The final simplified Hamiltonian contains only the variables from the original equations of motion. So the final result is that both the Hamiltonian as well as the dynamical (Dirac's) brackets are fully expressed by only the variables from the original equations of motion. All other variables are completely eliminated from the final version of the Hamiltonian formalism.

## II. The Hamilton equations

We will start with an $n$-dimensional second order system of equations. Let's assume that the equations of motion are of the second order, and they can be solved for the second derivatives, so they are given in the form ($i=1,\ldots,n$)

$$\ddot{x}_i = R_i(x, \dot{x}), \tag{2.1}$$

where the coordinates in the $n$-dimensional configuration space are $x = (x_1, x_2, \ldots, x_n)$, and the dot above a variable denotes the time derivative. Often equations of that kind are called "Newtonian".

Introducing velocity variables $v_i = \dot{x}_i$ , we can replace equations (2.1) by first order equations ($i = 1, \ldots, n$)

$$\begin{aligned} \dot{x}_i &= v_i \\ \dot{v}_i &= R_i(x, v), \end{aligned} \tag{2.2}$$

Arguably, the simplest physical system of the (2.2) kind is an $n$-dimensional free particle of unitary mass. Its equations of motion are ($i = 1, \ldots, n$)

$$\begin{aligned} \dot{y}_i &= w_i \\ \dot{w}_i &= 0, \end{aligned} \tag{2.3}$$

where $y = (y_1, y_2, \ldots, y_n)$ are coordinates in $n$-dimensional configuration space, and $w = (w_1, w_2, \ldots, w_n)$ are the velocities.

By using a flow box theorem we can show that locally there exists a change of variables between these two systems ($i = 1, \ldots, n$)

$$\begin{aligned} x_i &= f_i(y, w) \\ v_i &= g_i(y, w) \end{aligned} \tag{2.4}$$

and its inverse

$$\begin{aligned} y_i &= k_i(x, v) \\ w_i &= s_i(x, y), \end{aligned} \tag{2.5}$$

and these changes of variables are such that equations (2.2), when expressed in variables $(y, w)$, become (2.3), and equations (2.3), when expressed in variables $(x, v)$, become (2.2).

This also means that the time derivatives of the equations (2.4), when expressed by the time derivatives of $(y, w)$ and compared with the equations (2.2), give $(i = 1, \ldots, n)$

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{\partial f_i(y,w)}{\partial y_j} \cdot \dot{y}_j + \sum_{j=1}^{n} \frac{\partial f_i(y,w)}{\partial w_j} \cdot \dot{w}_j = v_i \\
&\sum_{j=1}^{n} \frac{\partial g_i(y,w)}{\partial y_j} \cdot \dot{y}_j + \sum_{j=1}^{n} \frac{\partial g_i(y,w)}{\partial w_j} \cdot \dot{w}_j = R_i(x,v).
\end{aligned} \tag{2.6}$$

Now using (2.2), we get $(i = 1, \ldots, n)$

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{\partial f_i(y,w)}{\partial y_j} \cdot w_j = v_i \\
&\sum_{j=1}^{n} \frac{\partial g_i(y,w)}{\partial y_j} \cdot w_j = R_i(x,v).
\end{aligned} \tag{2.7}$$

Similarly the time derivatives of the equations (2.5), when expressed by the time derivatives of $(x, v)$ and compared with the equations (2.3), give $(i = 1, \ldots, n)$

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{\partial k_i(x,v)}{\partial x_j} \cdot \dot{x}_j + \sum_{j=1}^{n} \frac{\partial k_i(x,v)}{\partial v_j} \cdot \dot{v}_j = w_i \\
&\sum_{j=1}^{n} \frac{\partial s_i(x,v)}{\partial x_j} \cdot \dot{x}_j + \sum_{j=1}^{n} \frac{\partial s_i(x,v)}{\partial v_j} \cdot \dot{v}_j = 0.
\end{aligned} \tag{2.8}$$

Now using (2.3), we get $(i = 1, \ldots, n)$

$$\begin{aligned}
&\sum_{j=1}^{n} \frac{\partial k_i(x,v)}{\partial x_j} \cdot v_j + \sum_{j=1}^{n} \frac{\partial k_i(x,v)}{\partial v_j} \cdot R_j(x,v) = w_i \\
&\sum_{j=1}^{n} \frac{\partial s_i(x,v)}{\partial x_j} \cdot v_j + \sum_{j=1}^{n} \frac{\partial s_i(x,v)}{\partial v_j} \cdot R_j(x,v) = 0.
\end{aligned} \tag{2.9}$$

In our previous work [8], we proposed the Lagrangian to be given by

$$L=\sum_{j=1}^{n}\frac{w_j^{\;2}}{2}+\sum_{j=1}^{n}\lambda_j(\dot{y}_j-w_j)+\sum_{j=1}^{n}\mu_j(x_j-f_j(y,w))+\sum_{j=1}^{n}\eta_j(v_j-g_j(y,w)), \tag{2.10}$$

where the functions $f_j$ and $g_j$ are such as defined in (2.4) and variables $\lambda_j,\mu_j$ and $\eta_j$ are new variables, treated on equal footing with all other variables. We have shown there that the Euler-Lagrange equations of that Lagrangian reproduce the original equations of motion (2.2), while also eliminating all the variables that are not present in the equations (2.2).

We will now concentrate on the Euler-Lagrange equations for this Lagrangian, expressed by the canonical momenta. We will call these equations Hamiltonian, despite of the fact that the Hamiltonian will be calculated later, in section IV.

Using the canonical momenta, the Euler-Lagrange equations have the form $\dot{p}_q=\frac{\partial L}{\partial q}$. For the Lagrangian (2.10), we get ($i=1,\ldots,n$):

$$\dot{p}_{x_i}=\mu_i \tag{2.11i}$$

$$\dot{p}_{v_i}=\eta_i \tag{2.11ii}$$

$$\dot{p}_{y_i}=-\sum_{j=1}^{n}\mu_j\frac{\partial f_j}{\partial y_i}-\sum_{j=1}^{n}\eta_j\frac{\partial g_j}{\partial y_i} \tag{2.11iii}$$

$$\dot{p}_{w_i}=w_i-\lambda_i-\sum_{j=1}^{n}\mu_j\frac{\partial f_j}{\partial w_i}-\sum_{j=1}^{n}\eta_j\frac{\partial g_j}{\partial w_i} \tag{2.11iv}$$

$$\dot{p}_{\lambda_i}=\dot{y}_i-w_i \tag{2.11v}$$

$$\dot{p}_{\mu_i}=x_i-f_i(y,w) \tag{2.11vi}$$

$$\dot{p}_{\eta_i} = v_i - g_i(y,w). \qquad (2.11vii)$$

The canonical momenta in the above equations are defined the usual way, using $p_q = \frac{\partial L}{\partial \dot{q}}$. Specifically ($i = 1,\ldots,n$):

$$p_{x_i} = 0 \qquad (2.11viii)$$

$$p_{v_i} = 0 \qquad (2.11ix)$$

$$p_{y_i} = \lambda_i \qquad (2.11x)$$

$$p_{w_i} = 0 \qquad (2.11xi)$$

$$p_{\lambda_i} = 0 \qquad (2.11xii)$$

$$p_{\mu_i} = 0 \qquad (2.11xiii)$$

$$p_{\eta_i} = 0. \qquad (2.11xiv)$$

Interestingly, all canonical momenta above represent constraints (primary) in the Dirac's sense [10,11]. So none of them can be used to express time derivatives of the variables $(x,v,y,w,\lambda,\mu,\eta)$. This is quite different from typical Lagrangians, where all velocities, or at least some of them, can be expressed by the canonical momenta.

We now use the equations (2.11) and follow the Dirac [10,11] procedure. (Although we do not use the modified Hamiltonians that Dirac uses, we use the equations only.) The essence of this procedure is that all equations that are not differential equations are constraints, and since they are supposed to hold over time, their time derivatives must be equal to zero. This may produce time derivatives of the variables that had no time derivatives among the equations (2.11i)-(2.11vii), or may produce new (independent) constraints. If there are new constraints then the procedure is repeated until no new constraints appear. The result of this somewhat tedious procedure done on equations (2.11) is ($i = 1,\ldots,n$) :

$$\dot{x}_i = \sum_{j=1}^{n} \frac{\partial f_i}{\partial y_j} \cdot w_j \tag{2.12i}$$

$$\dot{p}_{x_i} = 0 \tag{2.12ii}$$

$$\dot{v}_i = \sum_{j=1}^{n} \frac{\partial g_i}{\partial y_j} \cdot w_j \tag{2.12iii}$$

$$\dot{p}_{v_i} = 0 \tag{2.12iv}$$

$$\dot{y}_i = w_i \tag{2.12v}$$

$$\dot{p}_{y_i} = 0 \tag{2.12vi}$$

$$\dot{w}_i = 0 \tag{2.12vii}$$

$$\dot{p}_{w_i} = 0 \tag{2.12viii}$$

$$\dot{\lambda}_i = 0 \tag{2.12ix}$$

$$\dot{p}_{\lambda_i} = 0 \tag{2.12x}$$

$$\dot{\mu}_i = 0 \tag{2.12xi}$$

$$\dot{p}_{\mu_i} = 0 \tag{2.12xii}$$

$$\dot{\eta}_i = 0 \tag{2.12xiii}$$

$$\dot{p}_{\eta_i} = 0 \tag{2.12xiv}$$

$$x_i - f_i(y,w) = 0 \quad (2.12xv)$$

$$v_i - g_i(y,w) = 0 \quad (2.12xvi)$$

$$p_{y_i} - w_i = 0 \quad (2.12xvii)$$

$$p_{w_i} = 0 \quad (2.12xviii)$$

$$p_{x_i} = 0 \quad (2.12xix)$$

$$p_{v_i} = 0 \quad (2.12xx)$$

$$w_i - \lambda_i = 0 \quad (2.12xxi)$$

$$p_{\lambda_i} = 0 \quad (2.12xxii)$$

$$\mu_i = 0 \quad (2.12xxiii)$$

$$p_{\mu_i} = 0 \quad (2.12xxiv)$$

$$\eta_i = 0 \quad (2.12xxv)$$

$$p_{\eta_i} = 0. \quad (2.12xxvi)$$

At this point all existing constraints are consistent with the equations of motion (meaning that time derivatives of constraints are equal to zero). Also, the equations above contain time derivatives of all variables $(x, v, y, w, \lambda, \mu, \eta)$ used in the formalism and time derivatives of their canonical momenta $(p_x, p_v, p_y, p_w, p_\lambda, p_\mu, p_\eta)$. This means that no gauges in the Dirac's [10,11] sense are needed.

Now we use properties (2.7) of the variable change (2.4) getting ($i = 1, \ldots, n$):

$$\dot{x}_i = v_i \quad (2.13i)$$

$$\dot{p}_{x_i} = 0 \quad (2.13ii)$$

$$\dot{v}_i = R_i(x,v) \quad (2.13iii)$$

$$\dot{p}_{v_i} = 0 \quad (2.13iv)$$

$$\dot{y}_i = w_i \quad (2.13v)$$

$$\dot{p}_{y_i} = 0 \quad (2.13vi)$$

$$\dot{w}_i = 0 \quad (2.13vii)$$

$$\dot{p}_{w_i} = 0 \quad (2.13viii)$$

$$\dot{\lambda}_i = 0 \quad (2.13ix)$$

$$\dot{p}_{\lambda_i} = 0 \quad (2.13x)$$

$$\dot{\mu}_i = 0 \quad (2.13xi)$$

$$\dot{p}_{\mu_i} = 0 \quad (2.13xii)$$

$$\dot{\eta}_i = 0 \quad (2.13xiii)$$

$$\dot{p}_{\eta_i} = 0 \quad (2.13xiv)$$

$$x_i - f_i(y,w) = 0 \quad (2.13xv)$$

$$v_i - g_i(y,w) = 0 \quad (2.13xvi)$$

$$p_{y_i} - w_i = 0 \quad (2.13xvii)$$

$$p_{w_i} = 0 \quad (2.13xviii)$$

$$p_{x_i} = 0 \quad (2.13xix)$$

$$p_{v_i} = 0 \quad (2.13xx)$$

$$w_i - \lambda_i = 0 \quad (2.13xxi)$$

$$p_{\lambda_i} = 0 \quad (2.13xxii)$$

$$\mu_i = 0 \quad (2.13xxiii)$$

$$p_{\mu_i} = 0 \quad (2.13xxiv)$$

$$\eta_i = 0 \quad (2.13xxv)$$

$$p_{\eta_i} = 0. \quad (2.13xxvi)$$

We may observe that equations (2.13i) and (2.13iii) reproduce the original equations of motion. We also observe that constraints (2.13xv)-(2.13xxvi) can be used to eliminate all variables that were not present in the original equations of motion (2.2), expressing them by the variables that were present there. The same happens with all canonical momenta of all these additional variables.

## III. The dynamical brackets

Let us give symbols to constraints (2.13xv)-(2.13xxvi) by defining ($i = 1,\ldots,n$):

$$(\varphi_1)_i = x_i - f_i(y,w) \quad (3.1i)$$

$$(\varphi_2)_i = v_i - g_i(y,w) \quad (3.1ii)$$

$$(\varphi_3)_i = p_{y_i} - w_i \quad (3.1iii)$$

$$(\varphi_4)_i = p_{w_i} \quad (3.1iv)$$

$$(\varphi_5)_i = p_{x_i} \quad (3.1v)$$

$$(\varphi_6)_i = p_{v_i} \tag{3.1vi}$$

$$(\varphi_7)_i = w_i - \lambda_i \tag{3.1vii}$$

$$(\varphi_8)_i = p_{\lambda_i} \tag{3.1viii}$$

$$(\varphi_9)_i = \mu_i \tag{3.1ix}$$

$$(\varphi_{10})_i = p_{\mu_i} \tag{3.1x}$$

$$(\varphi_{11})_i = \eta_i \tag{3.1xi}$$

$$(\varphi_{12})_i = p_{\eta_i}\,. \tag{3.1xii}$$

Since we treat all variables $(x,v,y,w,\lambda,\mu,\eta)$ on the same footing, as completely independent variables, the Poisson brackets of two functions $f$ and $g$ of these variables is defined, in a usual way, as:

$$\begin{aligned}\{f,g\} = {} & \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\cdot\frac{\partial g}{\partial p_{x_i}} - \frac{\partial f}{\partial p_{x_i}}\cdot\frac{\partial g}{\partial x_i}\right) + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial v_i}\cdot\frac{\partial g}{\partial p_{v_i}} - \frac{\partial f}{\partial p_{v_i}}\cdot\frac{\partial g}{\partial v_i}\right) + \\ & + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial y_i}\cdot\frac{\partial g}{\partial p_{y_i}} - \frac{\partial f}{\partial p_{y_i}}\cdot\frac{\partial g}{\partial y_i}\right) + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial w_i}\cdot\frac{\partial g}{\partial p_{w_i}} - \frac{\partial f}{\partial p_{w_i}}\cdot\frac{\partial g}{\partial w_i}\right) + \\ & + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial \lambda_i}\cdot\frac{\partial g}{\partial p_{\lambda_i}} - \frac{\partial f}{\partial p_{\lambda_i}}\cdot\frac{\partial g}{\partial \lambda_i}\right) + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial \mu_i}\cdot\frac{\partial g}{\partial p_{\mu_i}} - \frac{\partial f}{\partial p_{\mu_i}}\cdot\frac{\partial g}{\partial \mu_i}\right) + \\ & + \sum_{i=1}^{n}\left(\frac{\partial f}{\partial \eta_i}\cdot\frac{\partial g}{\partial p_{\eta_i}} - \frac{\partial f}{\partial p_{\eta_i}}\cdot\frac{\partial g}{\partial \eta_i}\right).\end{aligned} \tag{3.2}$$

Following Dirac we calculate the constraints' matrix, defined for the constraints (3.1), using the Poisson brackets (3.2) as $(m,n=1,\ldots,12)$, $(i,\,j=1,\ldots,n)$:

$$[C_{ij}]_{mn} = \{(\varphi_m)_i,(\varphi_n)_j\} \tag{3.3}$$

Direct calculation gives ($i, j = 1, \ldots, n$):

$$[C_{ij}]_{mn} =$$

$$\begin{bmatrix}
0 & 0 & -\partial f_i/\partial y_j & -\partial f_i/\partial w_j & \delta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\partial g_i/\partial y_j & -\partial g_i/\partial w_j & 0 & \delta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial f_j/\partial y_i & \partial g_j/\partial y_i & 0 & -\delta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\partial f_j/\partial w_i & \partial g_j/\partial w_i & \delta_{ij} & 0 & 0 & 0 & -\delta_{ij} & 0 & 0 & 0 & 0 & 0 \\
-\delta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\delta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta_{ij} & 0 & 0 & 0 & -\delta_{ij} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \delta_{ij} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_{ij} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_{ij} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_{ij} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_{ij} & 0
\end{bmatrix} \tag{3.4}$$

It is possible to explicitly calculate the inverse of the matrix above, and the result is:

$$[C^{-1}{}_{ij}]_{mn} =$$

$$\begin{bmatrix}
0 & 0 & 0 & 0 & -\delta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\delta_{ij} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \delta_{ij} & \partial f_j / \partial w_i & \partial g_j / \partial w_i & 0 & \delta_{ij} & 0 & 0 & 0 & 0 \\
0 & 0 & -\delta_{ij} & 0 & -\partial f_j / \partial y_i & -\partial g_j / \partial y_i & 0 & 0 & 0 & 0 & 0 & 0 \\
\delta_{ij} & 0 & -\partial f_i / \partial w_j & \partial f_i / \partial y_j & \sum\limits_k \left[ \frac{\partial f_i}{\partial y_k} \cdot \frac{\partial f_j}{\partial w_k} - \frac{\partial f_i}{\partial w_k} \cdot \frac{\partial f_j}{\partial y_k} \right] & \sum\limits_k \left[ \frac{\partial f_i}{\partial y_k} \cdot \frac{\partial g_j}{\partial w_k} - \frac{\partial f_i}{\partial w_k} \cdot \frac{\partial g_j}{\partial y_k} \right] & 0 & \partial f_i / \partial y_j & 0 & 0 & 0 & 0 \\
0 & \delta_{ij} & -\partial g_i / \partial w_j & \partial g_i / \partial y_j & \sum\limits_k \left[ \frac{\partial g_i}{\partial y_k} \cdot \frac{\partial f_j}{\partial w_k} - \frac{\partial g_i}{\partial w_k} \cdot \frac{\partial f_j}{\partial y_k} \right] & \sum\limits_k \left[ \frac{\partial g_i}{\partial y_k} \cdot \frac{\partial g_j}{\partial w_k} - \frac{\partial g_i}{\partial w_k} \cdot \frac{\partial g_j}{\partial y_k} \right] & 0 & \partial g_i / \partial y_j & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_{ij} & 0 & 0 & 0 & 0 \\
0 & 0 & -\delta_{ij} & 0 & -\partial f_j / \partial y_i & -\partial g_j / \partial y_i & -\delta_{ij} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_{ij} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_{ij} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_{ij} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \delta_{ij} & 0
\end{bmatrix}$$

(3.5)

The Dirac's brackets [10,11] of two functions $f$ and $g$ of variables $(x, v, y, w, \lambda, \mu, \eta)$ are defined using Poisson Brackets (3.2), constraints (3.1), and inverse matrix (3.5) as:

$$\{f, g\}_D = \{f, g\} - \sum_{i,j=1}^{n} \sum_{m,n=1}^{12} \{f, (\varphi_m)_i\} \cdot [C^{-1}{}_{ij}]_{mn} \cdot \{(\varphi_n)_j, g\} \tag{3.6}$$

Calculating Dirac's bracket for all variables $(x, v, y, w, \lambda, \mu, \eta)$ and their canonical momenta is simple, although somewhat tedious. We will give results for only the variables that appear in the original equations of motion (2.2) $(i, j = 1, \ldots, n)$:

$$\{x_i, x_j\}_D = -\sum_k \left[ \frac{\partial f_i}{\partial w_k} \cdot \frac{\partial f_j}{\partial y_k} - \frac{\partial f_i}{\partial y_k} \cdot \frac{\partial f_j}{\partial w_k} \right] \qquad (3.7i)$$

$$\{x_i, v_j\}_D = -\sum_k \left[ \frac{\partial f_i}{\partial w_k} \cdot \frac{\partial g_j}{\partial y_k} - \frac{\partial f_i}{\partial y_k} \cdot \frac{\partial g_j}{\partial w_k} \right] \qquad (3.7ii)$$

$$\{v_i, v_j\}_D = -\sum_k \left[ \frac{\partial g_i}{\partial w_k} \cdot \frac{\partial g_j}{\partial y_k} - \frac{\partial g_i}{\partial y_k} \cdot \frac{\partial g_j}{\partial w_k} \right]. \qquad (3.7iii)$$

We also have

$$\{x_i, w_j\}_D = \partial f_i / \partial y_j \qquad (3.8i)$$

and

$$\{v_i, w_j\}_D = \partial g_i / \partial y_j. \qquad (3.8ii)$$

**IV. The Hamiltonian equations**

The Hamiltonian is given in a usual way, using the Lagrangian (2.10), as:

$$H = \sum_{j=1}^{n} \left( p_{x_j} \cdot \dot{x}_j + p_{y_j} \cdot \dot{y}_j + p_{v_j} \cdot \dot{v}_j + p_{w_j} \cdot \dot{w}_j + p_{\lambda_j} \cdot \dot{\lambda}_j + p_{\mu_j} \cdot \dot{\mu}_j + p_{\eta_j} \cdot \dot{\eta}_j \right) +$$
$$- \sum_{j=1}^{n} \frac{{w_j}^2}{2} - \sum_{i=1}^{n} \lambda_j (\dot{y}_j - w_j) - \sum_{j=1}^{n} \mu_j (x_j - f_j(y,w)) - \sum_{j=1}^{n} \eta_j (v_j - g_j(y,w)).$$

(4.1)

Now we use the constraints (3.1) to simplify the Hamiltonian (4.1), getting

$$\widetilde{H}=\sum_{j=1}^{n}\left(0\cdot\dot{x}_j+w_j\cdot\dot{y}_j+0\cdot\dot{v}_j+0\cdot\dot{w}_j+0\cdot\dot{\lambda}_j+0\cdot\dot{\mu}_j+0\cdot\dot{\eta}_j\right)+$$

$$-\sum_{j=1}^{n}\frac{w_j^{\,2}}{2}-\sum_{j=1}^{n}w_j(\dot{y}_j-w_j)-\sum_{j=1}^{n}0\cdot(x_j-f_j(y,w))-\sum_{j=1}^{n}0\cdot(v_j-g_j(y,w)).\tag{4.2}$$

After simplification it becomes

$$\widetilde{H}=\sum_{j=1}^{n}\frac{w_j^{\,2}}{2}.\tag{4.3}$$

A direct check shows that ($i=1,\ldots,n$):

$$\{x_i,\widetilde{H}\}_D=v_i\tag{4.4}$$

and

$$\{v_i,\widetilde{H}\}_D=R_i(x,v).\tag{4.5}$$

So the original equations of motions (2.2) are reproduced by ($i=1,\ldots,n$):

$$\dot{x}_i=\{x_i,\widetilde{H}\}_D\tag{4.6}$$

$$\dot{v}_i=\{v_i,\widetilde{H}\}_D\ .\tag{4.7}$$

However, we want to use the variables $(x,v)$ only, so we invert the constraints (3.1i) and (3.1ii), and formulas (2.5) to express the Hamiltonian (4.3) in terms of variables $(x,v)$ as

$$H_D=\sum_{j=1}^{n}\frac{s_j^{\,2}(x,v)}{2}.\tag{4.8}$$

It is the essence of the Dirac's brackets that they allow modifications of the Hamiltonian by the constraints, if these constraints were used to define the Dirac's bracket. Therefore the equations

$$\dot{x}_i = \{x_i, H_D\}_D \tag{4.9}$$

$$\dot{v}_i = \{v_i, H_D\}_D \tag{4.10}$$

will also reproduce the original equations (2.2). At the same time, the Hamiltonian $H_D$ contains only the variables $(x, v)$ from the original equations of motion (2.2). All other variables were eliminated by the constraints that appeared spontaneously from the Lagrangian (2.10).

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