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Piotr W. Hebda
University of North Georgia, piotr.hebda@ung.edu

Beata A. Hebda
University of North Georgia, beata.hebda@ung.edu

An example of a Lagrangian for a non-holonomic system

Piotr W. Hebda,[a] Beata A. Hebda

Department of Mathematics, University of North Georgia, Oakwood, Georgia, 30566, USA

An adjustable two-mass-point Chaplygin Sleigh is used as an example of a non-holonomic system. Newtonian equations of motion based the assumption of zero virtual work done by constraints are calculated. A Lagrangian that reproduces these equations as its unmodified Euler-Lagrange equations is then explicitly given. The Lagrangian uses variables that are present in the Chaplygin Sleigh equations of motion, as well as some additional variables. Some of the Euler-Lagrange equations of that Lagrangian are non-differential. These non-differential equations automatically and completely reduce out all of these additional variables, so that only the variables that appear in the original equations of motion remain in the final dynamics of the system.

I. INTRODUCTION

While the Newtonian equations of motion seem to be physically more fundamental than the Lagrangian that produces these equations as its Euler-Lagrange equations, the Lagrangian is still of great interest, since it provides a natural framework for further study of the system. For example it is a starting point for calculating the Hamiltonian and the Poisson brackets structure. The problem of constructing a Lagrangian for given equations of motion has been therefore extensively studied, but it is still not completely resolved. [1)]

Quite often we study a mechanical system for which a Lagrangian is already known, but which is subsequently modified by imposing additional constraints. The constraints

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[a)] Author to whom correspondence should be addressed. Electronic mail: Piotr.Hebda@UNG.edu

modify the original equations of motion, and the modifications then lead to the need of modifications of the Lagrangian. Modifying the Lagrangian is quite simple if the constraints are of holonomic type (constraints that could be expressed by restricting the allowable positions of the system). In this case the new Lagrangian is obtained by adding the constraints, each constraint multiplied by its own so called Lagrange multiplier, to the original Lagrangian.[2)] In the case of non-holonomic constraints (these are constraints that involve velocities and cannot be reduced to restricting the positions only) the situation is not so simple. Adding constraints multiplied by the Lagrange multipliers to the original Lagrangian will produce equations of motion that are acceptable from mathematical point of view, but they are different from actual physical equations that result from such constraints. Specifically, in the case of non-holonomic constraints, the constraints forces resulting from the use of Lagrange multipliers do not satisfy the condition of zero virtual work, which is expected to be satisfied in real-world mechanics.[3)] Because of that fact the use of Lagrange multipliers is generally rejected in the case of non-holonomic constraints. A commonly accepted approach for such a case is not to modify a Lagrangian at all, but to obtain the Euler-Lagrange equations from the Lagrangian, and then modify these equations to include forces resulting from the constraints.[3)] However, as the resulting final equations of motion are not the usual Euler-Lagrange equations of a known Lagrangian anymore, some advantages of using the Lagrangian are lost.

Another approach to the non-holonomic constraints can be done by adapting the Bateman-Morse-Feshbach approach.[4,5)] In this approach a Lagrangian for an arbitrary system of equations of motion is constructed by Lagrangian being equal to sum of all equations of motion, each equation multiplied by a new variable. The method is somewhat analogous to Lagrange multipliers method, extended not only to equations representing constraints, but to all equations of the system. This approach results in getting correct equations of motion directly as the Euler-Lagrange equations, but it also creates additional non-physical variables that were not existing in the original equations of motion. The additional variables are then present in the Lagrange-Hamilton formalism that follows, and it is not clear how to interpret them. So this approach, while relatively simple, is not commonly accepted as a resolution of the Lagrangian construction problem.

In this work we show a Lagrangian for one specific example of a non-holonomic system using the explicit solutions of the equations of motion to construct the Lagrangian, rather than a combination of the kinetic and potential energies used in a typical process of getting a Lagrangian. On some level this is a quite satisfying approach, since one may claim that solutions are more fundamental objects than kinetic and potential energies - solutions of equations are directly observable and they always exists, while kinetic and potential energies are more abstract constructs, and in some cases may not exists at all. On the other hand, generalization of our approach to other examples may be problematic, because in many situations we do not have explicit solutions of the equations of motion, and in these cases we will not be able to get the Lagrangian explicitly, which may be a serious drawback. However, some preliminary results[6)] suggest that even in such cases we can prove the existence of a Lagrangian, which by itself is an interesting result.

Our approach produces the correct equations of motion directly from the Lagrangian, as the Euler-Lagrange equations, with no further modifications necessary. Similarly to the Bateman-Morse-Feshbach approach,[4,5)] we also use variables that do not appear in the original equations of motion. We avoid the basic difficulty of the Bateman-Morse-Feshbach approach though, because some of the Euler-Lagrange equations obtained from our Lagrangian are constrains rather than differential equations. The constraints then automatically eliminate all variables that do not appear in the original equations of motion, while leaving the original equations of motion intact.

The organization of our presentation is as follows:

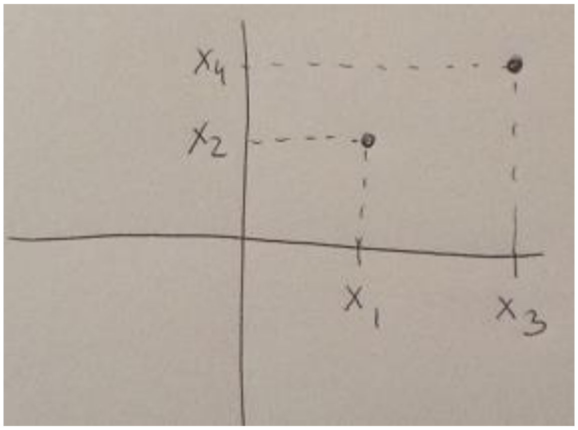
In section II, we define the adjustable two-mass-point Chaplygin Sleigh mechanical system, we show that it is a non-holonomic system, and we derive its equations of motion.

In section III, we present the proposed Lagrangian and we derive and simplify its Euler-Lagrange equations. We show that some of the equations are identical to the equations of motion for the adjustable two-mass-point Chaplygin Sleigh presented in section II. We also show that other Euler-Lagrange equations are constraints or time derivatives of the constraints, and that these constraints eliminate all the additional variables used to create the Lagrangian, leaving in the final dynamics only the variables that were present in the original equations of motion.

In section IV, we comment on a possibility to use Dirac's Theory of Constraints[7,8] to obtain the Hamiltonian formalism for our Lagrangian.

II. THE ADJUSTABLE TWO-MASS-POINT CHAPLYGIN SLEIGH

Physically our mechanical system will be made of two particles, each moving freely in two dimensions, and each having a unitary mass. Their position will be given by the usual variables (x_1, x_2, x_3, x_4), with the variables (x_1, x_2) describing the first particle, and variables (x_3, x_4) describing the second.



Using the time derivatives (v_1, v_2, v_3, v_4) of the variables (x_1, x_2, x_3, x_4) produces the following equations of motion, still unmodified by the constraints, as:

$$\dot{v}_1 = 0 \quad \text{(1i)}$$

$$\dot{v}_2 = 0 \quad \text{(1ii)}$$

$$\dot{v}_3 = 0 \quad \text{(1iii)}$$

$$\dot{v}_4 = 0 \quad \text{(1iv)}$$

$$\dot{x}_1 = v_1 \quad \text{(1v)}$$

$$\dot{x}_2 = v_2 \quad \text{(1fvi)}$$

$$\dot{x}_3 = v_3 \quad \text{(1vii)}$$

$$\dot{x}_4 = v_4, \quad \text{(1viii)}$$

where a dot above a variable means the time derivative.

To obtain an adjustable two-mass-point Chaplygin Sleigh from this free system, we impose two constraints:

$$(v_3 - v_1)(x_3 - x_1) + (v_4 - v_2)(x_4 - x_2) = 0 \quad \text{(2i)}$$

$$(v_1 + v_3)(x_4 - x_2) - (v_2 + v_4)(x_3 - x_1) = 0, \qquad \text{(2ii)}$$

that are supposed to be satisfied by all solutions of the equations of motion.

The constraint (2i) is holonomic, since it is obtained by taking a time derivative of

$$(x_3 - x_1)^2 + (x_4 - x_2)^2 = r^2.$$

r may vary with different initial positions of the particles, but the constraint (2i) assures that it remains constant during the motion. The possibility of having different values of r is the reason for calling our Chaplygin Sleigh "adjustable".

The constraint (2ii) is non-holonomic. It can be interpreted as the velocity $(\frac{v_1+v_3}{2}, \frac{v_2+v_4}{2})$ of the center of mass $(\frac{x_1+x_3}{2}, \frac{x_2+x_4}{2})$ being parallel to the vector $(x_3 - x_1, x_4 - x_2)$ which is starting at the first particle, and ending at the second. The vector $(\frac{v_1+v_3}{2}, \frac{v_2+v_4}{2})$ is parallel to $(x_3 - x_1, x_4 - x_2)$, since the constraint (2ii) says it is perpendicular to $(x_4 - x_2, -(x_3 - x_1))$.

This constraint (2ii) is non-holonomic since it allows a rotation of the particles around its center of mass, and also it allows a translation along the vector starting at the first particle and ending at the second. Combinations of these rotations and translations allow to reach all possible positions of the particles, once the distance between the particles is established by the constraint (2i). Therefore (2ii) is not imposing any restrictions on the possible positions of the system, while imposing restrictions on possible velocities.

The constraints (2) may be written in standard form as

$$\sum_{k=1}^{4} a_{jk}(x_1, x_2, x_3, x_4) \cdot v_k = 0, \qquad (3)$$

where $j = 1$ represents the constraint (2i), and $j = 2$ represents the constraint (2ii). Direct comparison then gives:

$$a_{11} = x_1 - x_3 \tag{4i}$$

$$a_{12} = x_2 - x_4 \tag{4ii}$$

$$a_{13} = x_3 - x_1 \tag{4iii}$$

$$a_{14} = x_4 - x_2 \tag{4iv}$$

$$a_{21} = x_4 - x_2 \tag{4v}$$

$$a_{22} = x_1 - x_3 \tag{4vi}$$

$$a_{23} = x_4 - x_2 \tag{4vii}$$

$$a_{24} = x_1 - x_3 \tag{4viii}$$

We assume that the forces of the constraint are such that they do zero work during instantaneous virtual displacements. It can be shown[3)] that from this assumption we get the constraint's forces to be

$$F_i = \sum_{j=1}^{2} \lambda_j a_{ji} \, ,$$

These forces then modify the equations of motion (1), giving

$$\dot{v}_1 = \lambda_1 (x_1 - x_3) + \lambda_2 (x_4 - x_2) \tag{5i}$$

$$\dot{v}_2 = \lambda_1 (x_2 - x_4) + \lambda_2 (x_1 - x_3) \tag{5ii}$$

$$\dot{v}_3 = \lambda_1 (x_3 - x_1) + \lambda_2 (x_4 - x_2) \tag{5iii}$$

$$\dot{v}_4 = \lambda_1 (x_4 - x_2) + \lambda_2 (x_1 - x_3) \tag{5iv}$$

$$\dot{x}_1 = v_1 \tag{5v}$$

$$\dot{x}_2 = v_2 \tag{5vi}$$

$$\dot{x}_3 = v_3 \tag{5vii}$$

$$\dot{x}_4 = v_4, \tag{5viii}$$

The constraints (2) must be preserved in time. So their time derivatives must be zero. This fact and the equations of motion (5) give us, after somewhat tedious calculations, the following expressions for λ_1 and λ_2:

$$\lambda_1 = -\frac{(v_3 - v_1)^2 + (v_4 - v_2)^2}{2(x_3 - x_1)^2 + 2(x_4 - x_2)^2}$$

$$\lambda_2 = \frac{v_3 v_2 - v_4 v_1}{(x_4 - x_2)^2 + (x_3 - x_1)^2}$$

Substituting that into the equations o motion (5), after some simplifications, gives:

$$\dot{v}_1 = \frac{(v_3 - v_1)^2 + (v_4 - v_2)^2}{2(x_3 - x_1)^2 + 2(x_4 - x_2)^2} \times (x_3 - x_1) + \frac{v_3 v_2 - v_4 v_1}{(x_4 - x_2)^2 + (x_3 - x_1)^2} \times (x_4 - x_2) \tag{6i}$$

$$\dot{v}_2 = \frac{(v_3 - v_1)^2 + (v_4 - v_2)^2}{2(x_3 - x_1)^2 + 2(x_4 - x_2)^2} \times (x_4 - x_2) - \frac{v_3 v_2 - v_4 v_1}{(x_4 - x_2)^2 + (x_3 - x_1)^2} \times (x_3 - x_1) \tag{6ii}$$

$$\dot{v}_3 = -\frac{(v_3 - v_1)^2 + (v_4 - v_2)^2}{2(x_3 - x_1)^2 + 2(x_4 - x_2)^2} \times (x_3 - x_1) + \frac{v_3 v_2 - v_4 v_1}{(x_4 - x_2)^2 + (x_3 - x_1)^2} \times (x_4 - x_2) \tag{6iii}$$

$$\dot{v}_4 = -\frac{(v_3 - v_1)^2 + (v_4 - v_2)^2}{2(x_3 - x_1)^2 + 2(x_4 - x_2)^2} \times (x_4 - x_2) - \frac{v_3 v_2 - v_4 v_1}{(x_4 - x_2)^2 + (x_3 - x_1)^2} \times (x_3 - x_1) \tag{6iv}$$

$$\dot{x}_1 = v_1 \tag{6v}$$

$$\dot{x}_2 = v_2 \tag{6vi}$$

$$\dot{x}_3 = v_3 \tag{6vii}$$

$$\dot{x}_4 = v_4, \tag{6viii}$$

$$(v_3 - v_1)(x_3 - x_1) + (v_4 - v_2)(x_4 - x_2) = 0 \tag{6ix}$$

$$(v_1 + v_3)(x_4 - x_2) - (v_2 + v_4)(x_3 - x_1) = 0 \tag{6x}$$

The last two equations are constraints. The equations (6) are the final equations of motion of the adjustable two-mass-point Chaplygin Sleigh.

III. THE LAGRANGIAN AND EULER-LAGRANGE EQUATIONS

Consider the Lagrangian

$$
\begin{aligned}
L = & \frac{w_1^{\,2}}{2} + \frac{w_2^{\,2}}{2} + \frac{w_3^{\,2}}{2} + \lambda_1(\dot{y}_1 - w_1) + \lambda_2(\dot{y}_2 - w_2) + \lambda_3(\dot{y}_3 - w_3) + \\
& + \mu_1[x_1 - w_3 \sin(y_1) - y_2 + w_2 \cdot \frac{y_1}{w_1} + w_2 \cos(y_1)] + \\
& + \mu_2[x_2 + w_3 \cos(y_1) - y_3 + w_3 \cdot \frac{y_1}{w_1} + w_2 \sin(y_1)] + \\
& + \mu_3[x_3 - w_3 \sin(y_1) - y_2 + w_2 \cdot \frac{y_1}{w_1} - w_2 \cos(y_1)] + \\
& + \mu_4[x_4 + w_3 \cos(y_1) - y_3 + w_3 \cdot \frac{y_1}{w_1} - w_2 \sin(y_1)] + \\
& + \eta_1[v_1 - w_1 w_3 \cos(y_1) - w_1 w_2 \sin(y_1)] + \\
& + \eta_2[v_2 - w_1 w_3 \sin(y_1) + w_1 w_2 \cos(y_1)] + \\
& + \eta_3[v_3 - w_1 w_3 \cos(y_1) + w_1 w_2 \sin(y_1)] + \\
& + \eta_4[v_4 - w_1 w_3 \sin(y_1) - w_1 w_2 \cos(y_1)] \,.
\end{aligned} \tag{7}
$$

In the Lagrangian (7) all the 25 variables, namely

$$(x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4, y_1, y_2, y_3, w_1, w_2, w_3, \lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \mu_4, \eta_1, \eta_2, \eta_3, \eta_4) \tag{8}$$

are treated on equal footing, as independent variables. (The specific formula for this Lagrangian was obtained using the procedure described by authors in a separate work.[6)])

To obtain the equations of motion from the Lagrangian (7), we use the standard Euler – Lagrange equations in the form $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$, where $q_i,\ i = 1,...,25$ represents all 25 variables (8). We obtain 25 equations:

$$\begin{aligned}\dot{\lambda}_1 = {} & \mu_1[-w_3\cos(y_1) + \frac{w_2}{w_1} - w_2\sin(y_1)] + \\ & + \mu_2[-w_3\sin(y_1) + \frac{w_3}{w_1} + w_2\cos(y_1)] + \\ & + \mu_3[-w_3\cos(y_1) + \frac{w_2}{w_1} + w_2\sin(y_1)] + \\ & + \mu_4[-w_3\sin(y_1) + \frac{w_3}{w_1} - w_2\cos(y_1)] + \\ & + \eta_1[w_1 w_3\sin(y_1) - w_1 w_2\cos(y_1)] + \\ & + \eta_2[-w_1 w_3\cos(y_1) - w_1 w_2\sin(y_1)] + \\ & + \eta_3[w_1 w_3\sin(y_1) + w_1 w_2\cos(y_1)] + \\ & + \eta_4[-w_1 w_3\cos(y_1) + w_1 w_2\sin(y_1)]\end{aligned} \qquad (9i)$$

$$\dot{\lambda}_2 = -\mu_1 - \mu_3 \qquad (9ii)$$

$$\dot{\lambda}_3 = -\mu_2 - \mu_4 \qquad (9iii)$$

$$\mu_1 = 0 \qquad (9iv)$$

$$\mu_2 = 0 \qquad (9v)$$

$$\mu_3 = 0 \qquad (9vi)$$

$$\mu_4 = 0 \qquad (9vii)$$

$$\eta_1 = 0 \qquad (9viii)$$

$$\eta_2 = 0 \qquad (9ix)$$

$$\eta_3 = 0 \qquad (9x)$$

$$\eta_4 = 0 \quad (9xi)$$

$$w_1 = \lambda_1 + \mu_1 \cdot \frac{w_2 y_1}{w_1^{\ 2}} + \mu_2 \cdot \frac{w_3 y_1}{w_1^{\ 2}} + \mu_3 \cdot \frac{w_2 y_1}{w_1^{\ 2}} + \mu_4 \cdot \frac{w_3 y_1}{w_1^{\ 2}} +$$
$$+ \eta_1 \cdot [w_3 \cos(y_1) + w_2 \sin(y_1)] + \eta_2 \cdot [w_3 \sin(y_1) - w_2 \cos(y_1)] + \quad (9xii)$$
$$+ \eta_3 \cdot [w_3 \cos(y_1) - w_2 \sin(y_1)] + \eta_4 \cdot [w_3 \sin(y_1) + w_2 \cos(y_1)]$$

$$w_2 = \lambda_2 - \mu_1 \cdot [\frac{y_1}{w_1} + \cos(y_1)] - \mu_2 \cdot \sin(y_1) - \mu_3 \cdot [\frac{y_1}{w_1} - \cos(y_1)] + \mu_4 \cdot \sin(y_1) + \quad (9xiii)$$
$$+ \eta_1 \cdot [w_1 \sin(y_1)] - \eta_2 \cdot [w_1 \cos(y_1)] - \eta_3 [w_1 \sin(y_1)] + \eta_4 [w_1 \cos(y_1)]$$

$$w_3 = \lambda_3 + \mu_1 \cdot \sin(y_1) - \mu_2 \cdot [\frac{y_1}{w_1} + \cos(y_1)] + \mu_3 \cdot \sin(y_1) - \mu_4 \cdot [\frac{y_1}{w_1} + \cos(y_1)] + \quad (9xiv)$$
$$+ \eta_1 \cdot [w_1 \cos(y_1)] + \eta_2 \cdot [w_1 \sin(y_1)] + \eta_3 [w_1 \cos(y_1)] + \eta_4 [w_1 \sin(y_1)]$$

$$\dot{y}_1 = w_1 \quad (9xv)$$

$$\dot{y}_2 = w_2 \quad (9xvi)$$

$$\dot{y}_3 = w_3 \quad (9xvii)$$

$$x_1 = w_3 \sin(y_1) + y_2 - w_2 \cdot \frac{y_1}{w_1} - w_2 \cos(y_1) \quad (9xviii)$$

$$x_2 = -w_3 \cos(y_1) + y_3 - w_3 \cdot \frac{y_1}{w_1} - w_2 \sin(y_1) \quad (9xix)$$

$$x_3 = w_3 \sin(y_1) + y_2 - w_2 \cdot \frac{y_1}{w_1} + w_2 \cos(y_1) \quad (9xx)$$

$$x_4 = -w_3 \cos(y_1) + y_3 - w_3 \cdot \frac{y_1}{w_1} + w_2 \sin(y_1) \quad (9xxi)$$

$$v_1 = w_1 w_3 \cos(y_1) + w_1 w_2 \sin(y_1) \quad (9xxii)$$

$$v_2 = w_1 w_3 \sin(y_1) - w_1 w_2 \cos(y_1) \quad (9xxiii)$$

$$v_3 = w_1 w_3 \cos(y_1) - w_1 w_2 \sin(y_1) \quad (9xxiv)$$

$$v_4 = w_1 w_3 \sin(y_1) + w_1 w_2 \cos(y_1) \tag{9xxv}$$

Using equations (9iv) – (9xi) to simplify equations (9i) – (9iii) and (9xii) – (9xiv) and rearranging the order of equations, we get:

$$\dot{\lambda}_1 = 0 \tag{10i}$$

$$\dot{\lambda}_2 = 0 \tag{10ii}$$

$$\dot{\lambda}_3 = 0 \tag{10iii}$$

$$\dot{y}_1 = w_1 \tag{10iv}$$

$$\dot{y}_2 = w_2 \tag{10v}$$

$$\dot{y}_3 = w_3 \tag{10vi}$$

$$w_1 = \lambda_1 \tag{10vii}$$

$$w_2 = \lambda_2 \tag{10viii}$$

$$w_3 = \lambda_3 \tag{10xix}$$

$$x_1 = w_3 \sin(y_1) + y_2 - w_2 \cdot \frac{y_1}{w_1} - w_2 \cos(y_1) \tag{10x}$$

$$x_2 = -w_3 \cos(y_1) + y_3 - w_3 \cdot \frac{y_1}{w_1} - w_2 \sin(y_1) \tag{10xi}$$

$$x_3 = w_3 \sin(y_1) + y_2 - w_2 \cdot \frac{y_1}{w_1} + w_2 \cos(y_1) \tag{10xii}$$

$$x_4 = -w_3 \cos(y_1) + y_3 - w_3 \cdot \frac{y_1}{w_1} + w_2 \sin(y_1) \tag{10xiii}$$

$$v_1 = w_1 w_3 \cos(y_1) + w_1 w_2 \sin(y_1) \tag{10xiv}$$

$$v_2 = w_1 w_3 \sin(y_1) - w_1 w_2 \cos(y_1) \tag{10xv}$$

$$v_3 = w_1 w_3 \cos(y_1) - w_1 w_2 \sin(y_1) \quad (10xvi)$$

$$v_4 = w_1 w_3 \sin(y_1) + w_1 w_2 \cos(y_1) \quad (10xvii)$$

$$\mu_1 = 0 \quad (10xviii)$$

$$\mu_2 = 0 \quad (10xix)$$

$$\mu_3 = 0 \quad (10xx)$$

$$\mu_4 = 0 \quad (10xxi)$$

$$\eta_1 = 0 \quad (10xxii)$$

$$\eta_2 = 0 \quad (10xxiii)$$

$$\eta_3 = 0 \quad (10xxiv)$$

$$\eta_4 = 0 \quad (10xxv)$$

The equations (10i) – (10vi) give us time derivatives of the variables $(\lambda_1, \lambda_2, \lambda_3, y_1, y_2, y_3)$. The equations (10vii) – (10xxv) are constraints. Constraints must hold as time progresses, so for each constraint time derivatives of both sides must be equal. In general taking time derivatives of existing constraints may create new constraints and/or give time derivatives of the variables that were not included in the earlier equation. In case of equations (10) taking time derivatives of constraints creates no new constraints. Instead it gives us time derivatives for all the variables that had no time derivatives in equations (10), namely time derivatives of the variables $(w_1, w_2, w_3, x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4, \mu_1, \mu_2, \mu_3, \mu_4, \eta_1, \eta_2, \eta_3, \eta_4)$.

This somewhat tedious process produces the following formulas for the time derivatives of all the 25 variables of the Lagrangian (7):

$$\dot{v}_1 = -w_1^2 w_3 \sin(y_1) + w_1^2 w_2 \cos(y_1) \quad (11i)$$

$$\dot{v}_2 = w_1^2 w_3 \cos(y_1) + w_1^2 w_2 \sin(y_1) \quad (11ii)$$

$$\dot{v}_3 = -w_1^2 w_3 \sin(y_1) - w_1^2 w_2 \cos(y_1) \quad (11iii)$$

$$\dot{v}_4 = w_1^2 w_3 \cos(y_1) - w_1^2 w_2 \sin(y_1) \quad (11iv)$$

$$\dot{x}_1 = v_1 \quad (11v)$$

$$\dot{x}_2 = v_2 \quad (11vi)$$

$$\dot{x}_3 = v_3 \quad (11vii)$$

$$\dot{x}_4 = v_4 \quad (11viii)$$

$$\dot{y}_1 = w_1 \quad (11ix)$$

$$\dot{y}_2 = w_2 \quad (11x)$$

$$\dot{y}_3 = w_3 \quad (11xi)$$

$$\dot{w}_1 = 0 \quad (11xii)$$

$$\dot{w}_2 = 0 \quad (11xiii)$$

$$\dot{w}_3 = 0 \quad (11xiv)$$

$$\dot{\lambda}_1 = 0 \quad (11xv)$$

$$\dot{\lambda}_2 = 0 \quad (11xvi)$$

$$\dot{\lambda}_3 = 0 \quad (11xvii)$$

$$\dot{\mu}_1 = 0 \quad (11xviii)$$

$$\dot{\mu}_2 = 0 \quad (11xix)$$

$$\dot{\mu}_3 = 0 \quad (11xx)$$

$$\dot{\mu}_4 = 0 \quad (11xxi)$$

$$\dot{\eta}_1 = 0 \quad (11xxii)$$

$$\dot{\eta}_2 = 0 \quad (11xxiii)$$

$$\dot{\eta}_3 = 0 \quad (11xxiv)$$

$$\dot{\eta}_4 = 0 \quad (11xxv)$$

and the constraints

$$x_1 = w_3 \sin(y_1) + y_2 - w_2 \cdot \frac{y_1}{w_1} - w_2 \cos(y_1) \quad (11xvi)$$

$$x_2 = -w_3 \cos(y_1) + y_3 - w_3 \cdot \frac{y_1}{w_1} - w_2 \sin(y_1) \quad (11xvii)$$

$$x_3 = w_3 \sin(y_1) + y_2 - w_2 \cdot \frac{y_1}{w_1} + w_2 \cos(y_1) \quad (11xviii)$$

$$x_4 = -w_3 \cos(y_1) + y_3 - w_3 \cdot \frac{y_1}{w_1} + w_2 \sin(y_1) \quad (11xix)$$

$$v_1 = w_1 w_3 \cos(y_1) + w_1 w_2 \sin(y_1) \quad (11xx)$$

$$v_2 = w_1 w_3 \sin(y_1) - w_1 w_2 \cos(y_1) \quad (11xxi)$$

$$v_3 = w_1 w_3 \cos(y_1) - w_1 w_2 \sin(y_1) \quad (11xxii)$$

$$v_4 = w_1 w_3 \sin(y_1) + w_1 w_2 \cos(y_1) \quad (11xxiii)$$

$$w_1 = \lambda_1 \quad (11xxiv)$$

$$w_2 = \lambda_2 \tag{11xxv}$$

$$w_3 = \lambda_3 \tag{11xxvi}$$

$$\mu_1 = 0 \tag{11xxvii}$$

$$\mu_2 = 0 \tag{11xxviii}$$

$$\mu_3 = 0 \tag{11xxix}$$

$$\mu_4 = 0 \tag{11xxx}$$

$$\eta_1 = 0 \tag{11xxxi}$$

$$\eta_2 = 0 \tag{11xxxii}$$

$$\eta_3 = 0 \tag{11xxxiii}$$

$$\eta_4 = 0 \tag{11xxxiv}$$

By direct calculations, using the constraints (11xvi) - (11xxiii) we get

$$-\frac{(v_3-v_1)^2+(v_4-v_2)^2}{2(x_3-x_1)^2+2(x_4-x_2)^2}\times(x_1-x_3)+\frac{v_3v_2-v_4v_1}{(x_4-x_2)^2+(x_3-x_1)^2}\times(x_4-x_2)= \\ =-w_1^2\,w_3\sin(y_1)+w_1^2w_2\cos(y_1) \tag{12i}$$

$$-\frac{(v_3-v_1)^2+(v_4-v_2)^2}{2(x_3-x_1)^2+2(x_4-x_2)^2}\times(x_2-x_4)+\frac{v_3v_2-v_4v_1}{(x_4-x_2)^2+(x_3-x_1)^2}\times(x_1-x_3)= \\ =w_1^2w_3\cos(y_1)+w_1^2w_2\sin(y_1) \tag{12ii}$$

$$-\frac{(v_3-v_1)^2+(v_4-v_2)^2}{2(x_3-x_1)^2+2(x_4-x_2)^2}\times(x_3-x_1)+\frac{v_3v_2-v_4v_1}{(x_4-x_2)^2+(x_3-x_1)^2}\times(x_4-x_2)= \\ =-w_1^2\,w_3\sin(y_1)-w_1^2\,w_2\cos(y_1) \tag{12iii}$$

$$-\frac{(v_3-v_1)^2+(v_4-v_2)^2}{2(x_3-x_1)^2+2(x_4-x_2)^2}\times(x_4-x_2)+\frac{v_3v_2-v_4v_1}{(x_4-x_2)^2+(x_3-x_1)^2}\times(x_1-x_3)=$$

(12iv)

$$=w_1^2w_3\cos(y_1)-w_1^2w_2\sin(y_1)$$

$$(v_3-v_1)(x_3-x_1)+(v_4-v_2)(x_4-x_2)=0 \qquad (12v)$$

$$(v_1+v_3)(x_4-x_2)-(v_2+v_4)(x_3-x_1)=0 \qquad (12v)$$

If we now use the equations (12i) – (12iv) to replace the right sides of the equations (11i) – (11iv) and include equations (12iv) and (12v) with other equations, we obtain the following system of equations of motion:

$$\dot{v}_1=-\frac{(v_3-v_1)^2+(v_4-v_2)^2}{2(x_3-x_1)^2+2(x_4-x_2)^2}\times(x_1-x_3)+\frac{v_3v_2-v_4v_1}{(x_4-x_2)^2+(x_3-x_1)^2}\times(x_4-x_2) \qquad (12i)$$

$$\dot{v}_2=-\frac{(v_3-v_1)^2+(v_4-v_2)^2}{2(x_3-x_1)^2+2(x_4-x_2)^2}\times(x_2-x_4)+\frac{v_3v_2-v_4v_1}{(x_4-x_2)^2+(x_3-x_1)^2}\times(x_1-x_3) \qquad (12ii)$$

$$\dot{v}_3=-\frac{(v_3-v_1)^2+(v_4-v_2)^2}{2(x_3-x_1)^2+2(x_4-x_2)^2}\times(x_3-x_1)+\frac{v_3v_2-v_4v_1}{(x_4-x_2)^2+(x_3-x_1)^2}\times(x_4-x_2) \qquad (12iii)$$

$$\dot{v}_4=-\frac{(v_3-v_1)^2+(v_4-v_2)^2}{2(x_3-x_1)^2+2(x_4-x_2)^2}\times(x_4-x_2)+\frac{v_3v_2-v_4v_1}{(x_4-x_2)^2+(x_3-x_1)^2}\times(x_1-x_3) \qquad (12iv)$$

$$\dot{x}_1=v_1 \qquad (12v)$$

$$\dot{x}_2=v_2 \qquad (12vi)$$

$$\dot{x}_3=v_3 \qquad (12vii)$$

$$\dot{x}_4=v_4 \qquad (12viii)$$

with constraints

$$(v_3 - v_1)(x_3 - x_1) + (v_4 - v_2)(x_4 - x_2) = 0 \qquad \text{(12ix)}$$

$$(v_1 + v_3)(x_4 - x_2) - (v_2 + v_4)(x_3 - x_1) = 0 \qquad \text{(12x)}$$

$$v_1 = w_1 w_3 \cos(y_1) + w_1 w_2 \sin(y_1) \qquad \text{(13i)}$$

$$v_2 = w_1 w_3 \sin(y_1) - w_1 w_2 \cos(y_1) \qquad \text{(13ii)}$$

$$v_3 = w_1 w_3 \cos(y_1) - w_1 w_2 \sin(y_1) \qquad \text{(13iii)}$$

$$v_4 = w_1 w_3 \sin(y_1) + w_1 w_2 \cos(y_1) \qquad \text{(13iv)}$$

$$x_1 = w_3 \sin(y_1) + y_2 - w_2 \cdot \frac{y_1}{w_1} - w_2 \cos(y_1) \qquad \text{(13v)}$$

$$x_2 = -w_3 \cos(y_1) + y_3 - w_3 \cdot \frac{y_1}{w_1} - w_2 \sin(y_1) \qquad \text{(13vi)}$$

$$x_3 = w_3 \sin(y_1) + y_2 - w_2 \cdot \frac{y_1}{w_1} + w_2 \cos(y_1) \qquad \text{(13vii)}$$

$$x_4 = -w_3 \cos(y_1) + y_3 - w_3 \cdot \frac{y_1}{w_1} + w_2 \sin(y_1) \qquad \text{(13viii)}$$

$$\lambda_1 = w_1 \qquad \text{(13ix)}$$

$$\lambda_2 = w_3 \qquad \text{(13x)}$$

$$\lambda_3 = w_3 \qquad \text{(13xi)}$$

$$\mu_1 = 0 \qquad \text{(13xii)}$$

$$\mu_2 = 0 \qquad \text{(13xiii)}$$

$$\mu_3 = 0 \qquad \text{(13xiv)}$$

$$\mu_4 = 0 \qquad \text{(13xv)}$$

$$\eta_1 = 0 \qquad (13xvi)$$

$$\eta_2 = 0 \qquad (13xvii)$$

$$\eta_3 = 0 \qquad (13xviii)$$

$$\eta_4 = 0 \qquad (13xix)$$

$$\dot{y}_1 = w_1 \qquad (14i)$$

$$\dot{y}_2 = w_2 \qquad (14ii)$$

$$\dot{y}_3 = w_3 \qquad (14iii)$$

$$\dot{w}_1 = 0 \qquad (14iv)$$

$$\dot{w}_1 = 0 \qquad (14v)$$

$$\dot{w}_3 = 0 \qquad (14vi)$$

$$\dot{\lambda}_1 = 0 \qquad (14vii)$$

$$\dot{\lambda}_2 = 0 \qquad (14viii)$$

$$\dot{\lambda}_3 = 0 \qquad (14ix)$$

$$\dot{\mu}_1 = 0 \qquad (14x)$$

$$\dot{\mu}_2 = 0 \qquad (14xi)$$

$$\dot{\mu}_3 = 0 \qquad (14xii)$$

$$\dot{\mu}_4 = 0 \qquad (14xiii)$$

$$\dot{\eta}_1 = 0 \qquad (14xiv)$$

$$\dot{\eta}_2 = 0 \qquad (14xv)$$

$$\dot{\eta}_3 = 0 \qquad (14xvi)$$

$$\dot{\eta}_4 = 0 \qquad (14xvii)$$

Let us stress that the entire system of equations (12), (13), and (14) is completely equivalent to the system (9) of the Euler-Lagrange equations of the Lagrangian (7).

Let us now interpret the system of equations (12), (13), and (14). First, the constraints (13) may be interpreted as implicit definition of all the variables $(y_1, y_2, y_3, w_1, w_2, w_3, \lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \mu_4, \eta_1, \eta_2, \eta_3, \mu_4)$ that do not appear in the adjustable two-mass-point Chaplygin Sleigh (6), by the variables $(x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4)$ that do appear there.

Moreover, it can be shown that all time derivatives (14) of the variables not appearing in the Chaplygin Sleigh equations can be obtained directly by taking time derivatives of the constraints (13), and then using equations (12).

This means that the variables $(y_1, y_2, y_3, w_1, w_2, w_3, \lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \mu_4, \eta_1, \eta_2, \eta_3, \mu_4)$ are completely dependent of variables $(x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4)$. The former are defined by the latter, and time derivatives of the former are the results of these definitions and the time derivatives of the latter. Therefore the variables $(y_1, y_2, y_3, w_1, w_2, w_3, \lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \mu_4, \eta_1, \eta_2, \eta_3, \mu_4)$ are just redundant variables on the space described by the variables $(x_1, x_2, x_3, x_4, v_1, v_2, v_3, v_4)$.

Please also notice that the equations (12) are identical to equations of motion of the Chaplygin Sleigh (6) obtained in section II. Concluding, the Lagrangian (7) gives the correct equations of motion for the Chaplygin Sleigh, while at the same time completely eliminating the additional variables used for its construction from the final dynamics of the system.

V. A COMMENT ON A HAMILTONIAN

Since our Lagrangian is degenerate to the extreme, with no velocities expressible by the canonical momenta, the Dirac's Theory of Constraints[7,8)] is a natural choice for creating the Hamiltonian formalism. Some preliminary results[6)] suggest that it will be possible to explicitly calculate both the Hamiltonian and the Dirac's Brackets for the adjustable two-mass-point Chaplygin Sleigh shown in this work, and that Dirac's Brackets of all variables appearing in the Lagrangian (7), but not appearing in the equations of motion (6), as well as the canonical momenta of these variables, will be equal to zero with every function using any variable of the system.

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