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Local existence of Dynamically Allowed Brackets and a local existence of a Hamiltonian associated with these Brackets, for a given, possibly time dependent, N-dimensional Equations of Motion that may include Constraints

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Local existence of Dynamically Allowed Brackets and a local existence of a Hamiltonian associated with these Brackets, for a given, possibly time dependent, N-dimensional Equations of Motion, that may include Constraints

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Dynamically Allowed Brackets are defined for given equations of motion for an N-dimensional mechanical system. The equations of motion may be explicitly time dependent, and may include explicitly time dependent constraints. Local existence of the Dynamically Allowed Brackets is shown. The local existence of a Hamiltonian reproducing the given equations of motion with the use of these Dynamically Allowed Brackets is proven.

I. INTRODUCTION

Ever since in the year 1950 P.A.M. Dirac introduced modifications to the Poisson Brackets, later called the Dirac Brackets [1,2] , the idea of using modified Poisson Brackets has been widely accepted. So called Generalized Poisson Brackets [3] are dynamical brackets that are specified less strictly than the Poisson Brackets, allowing some freedom in choosing them and, among others, include the Dirac Brackets.

The work presented here started with an observation that if a system of the equations of motion is given, then not all Generalized Poisson Brackets will be equally suitable for working with these equations. Only a subset of Generalized Poisson Brackets will work, and this subset will be dependent on the equations of motion under the consideration.

The second observation essential for this work is that it is not always convenient to present the dynamical brackets using the usual positions-momentum coordinates, as it is customary

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. Even more, since we aim at defining the brackets at the level of the equations of motion, without the knowledge of the Lagrangian and/or Hamiltonian, we have no choice but to use position-velocity coordinates.

In this work we study a definition of a subset of all possible Generalized Poisson Brackets, choosing among them only these bracket systems that are in agreement with the given equations of motion. We call such brackets Dynamically Allowed Brackets, and we define them by an explicit and easy to verify reference to the equations of motion. Then we study properties of this class of brackets, concluding that they always allow the existence of a Hamiltonian. Conversely, it turns out that if a Hamiltonian exists, then the brackets that reproduce the equations of motion using that Hamiltonian, are automatically Dynamically Allowed Brackets.

We would like to stress that in our approach the brackets and the Hamiltonian are obtained directly from the equations, without the need of defining the Lagrangian or the canonical momenta first. We also prove the local existence of Dynamically Allowed Brackets for any smooth enough system of equations.

The organization of our presentation is as follows:

In section II, we specify the general form of the starting equations of motion that we are going to consider and their form after a simplification procedure.

In section III, we recall the definition of the Generalized Poisson Brackets and their basic properties.

In section IV, we define, for any given system of equations of motion, a subset of all possible Generalized Poisson Brackets that we call Dynamically Allowed Brackets. We present a simple way of verifying if given Generalized Poisson Brackets are Dynamically Allowed Brackets for a given set of equations of motion.

In section V, we show that for each given system of equations of motion and any system of their Dynamically Allowed Brackets there exists a function called a Hamiltonian, unique up to an additive function of time, that reproduces the given equations of motion via these Dynamically Allowed Brackets. And conversely, if a system of equations of motion is reproduced from a Hamiltonian by Generalized Poisson Brackets, then these specific Generalized Poisson Brackets are Dynamically Allowed Brackets for these equations of motion.

In section VI, we show that, in general, for a given system of equations of motion, not all Generalized Poisson Brackets are Dynamically Allowed Brackets for that system. This shows the need for restricting the set of all possible Generalized Poisson Brackets to the narrower set of the Dynamically Allowed Brackets.

In section VII, we show that for any smooth enough system of equations of motion at least one set of Dynamically Allowed Brackets exists. We also give some characterization of all systems of Dynamically Allowed Brackets for a given system of equations of motion.

In section VIII, we comment on the possibility of using the Dynamically Allowed Brackets for a quantization process that would start from the classical equations of motion and end up with a quantized system, without ever using canonical coordinates nor a Hamiltonian.

II. INITIAL EQUATIONS OF MOTION AND THEIR STANDARD FORM

Let's say that a mechanical system has configuration space of dimension N described by generalized coordinates $q_i, i = 1,...,N$, called "positions" and, if convenient, denoted by (q). Let the equations of motion of this system be given as:

$$A_k(\ddot{q}_i, \dot{q}_i, q_i, t) = 0, \quad i = 1,...,N, \quad k = 1,...,M, \qquad (1)$$

where the dot over a variable means the derivative with respect to time t, and two dots mean the second derivative with respect to t. The number of equations M is not correlated in any way with the dimension N of the configuration space. The second derivative with respect to time does not have to show up explicitly in every equation, or even any equation in (1). In general, some or all equations may contain only the first time derivative(s), or even no time derivatives at all.

We reduce the order of the equations (1) by introducing the variables $v_i, \quad i=1,...,N$ which we will call "velocities" and, when convenient, denote by (v). Their equations of motion are, by definition:

$$v_i = \dot{q}_i, \quad i=1,...,N. \tag{2}$$

Using velocities we can rewrite equations of motion (1) as

$$\begin{aligned} &\dot{q}_i = v_i, \\ &A_k(\dot{v}_i, v_i, q_i, t) = 0, \quad i=1,...,N, \quad k=1,...,M. \end{aligned} \tag{3}$$

Equations (3) can be modified to a standardized form. Without going into details, we solve (2) for as many time derivatives of velocities as possible, and get more time derivatives of velocities by taking time derivatives of some equations.[3)] We continue the process until it stops producing independent equations for either the coordinates (q,v) or time derivatives of (v). If at the end of the process some time derivatives of some $v's$ are not solved for, we arbitrarily set the constraints on respective $q's$, so that the velocities $v's$ and their time derivatives become solved for. We can interpret this arbitrary choice as just one representative of the class of possible solutions, which are arbitrary, since these derivatives are missing.

At the end of this process the system of equations (1) is rewritten as:

$$\dot{q}_i = v_i,$$
$$\dot{v}_i = F_i(v,q,t) \qquad i = 1,...,N \tag{4}$$

$$C_u(v,q,t) = 0 \qquad u = 1,...,U \le 2N\,. \tag{5}$$

where equations (5) are all equations that do not contain any time derivatives. We call them constraints. They restrict the $q's$ and $v's$ possible for the solutions of the system of equations (1). They are stable under the time evolutions allowed under the equations (1) – if they were not stable, they would produce more constraints, and we assumed that the process of creating new constraints has finished.

At this point it is convenient to change the notation. We will introduce $z_i = q_i \;\; i = 1,...,N$, and $z_i = v_{i-N} \;\; i = N+1,...,2N$. If convenient, we will use (z) when referring to these coordinates. The equations (4) and (5) can be written in the form:

$$\dot{z}_i = E_i(z,t) \qquad i = 1,...,2N \tag{6}$$

$$C_u(z,t) = 0 \qquad u = 1,...,U \le 2N\,. \tag{7}$$

The equations (6) and (7) will be referred to as standard form of the equations of motion (1).

III. GENERALIZED POISSON BRACKETS

In this section we will recall the definition and basic properties of the Generalized Poisson Brackets. Let $P_{ij}(z,t), \quad i,j = 1,...2N$ be a matrix field in which, for every (z,t), the matrices satisfy the conditions:

$$\det P_{ij}(z,t) \ne 0 \qquad \text{(invertibility)} \tag{8}$$

$$P_{km}(z,t) = -P_{mk}(z,t) \qquad \text{(antisymmetry)} \tag{9}$$

$$P_{na}(z,t)\cdot\frac{\partial P_{bc}(z,t)}{\partial z_n}+P_{nb}(z,t)\cdot\frac{\partial P_{ca}(z,t)}{\partial z_n}+P_{nc}(z,t)\cdot\frac{\partial P_{ab}(z,t)}{\partial z_n}=0 \quad \text{(Jacobi condition).} \qquad (10)$$

In equation (10) we used the Einstein's summation convention. We will also use it in the remainder of this paper.

Let $f(z,t)$ and $g(z,t)$ be two smooth enough functions. Choose one specific matrix field satisfying the conditions (8) – (10) above. Then the Generalized Poisson Brackets (associated with that matrix field) is an operator $\{\ \ ,\ \ \}$, which produces another function of (z,t), given by

$$\{f,g\}(z,t)=\frac{\partial f(z,t)}{\partial z_i}\cdot P_{ij}(z,t)\cdot\frac{\partial g(z,t)}{\partial z_j}. \qquad (11)$$

Notice that in the definition of the Generalized Poisson Brackets, the brackets are defined completely separately at every moment of time t. There are no conditions whatsoever in the definition that would connect the $P_{ij}(z,t)$ from one moment of time to another. The time evolution of $P_{ij}(z,t)$ is completely arbitrary, as long as for each moment of time they satisfy the conditions (8) – (10).

It is easy to check that the Generalized Poisson Brackets have the following properties:

1) $\{f,g\}=-\{g,f\}$ (antisymmetry) (12)

2) $\{c_1 f_1+c_2 f_2,g\}=c_1\{f_1,g\}+c_2\{f_2,g\}$ (linearity), (13)

where c_1 and c_2 are numbers (this means they do not depend on (z), but may depend on t).

3) $\{f_1\cdot f_2,g\}=f_1\cdot\{f_2,g\}+\{f_1,g\}\cdot f_2$ (product rule), (14)

4) $\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=0$ (Jacobi identity). (15)

5) Only functions that do not depend explicitly on the coordinates (z) have Generalized Poisson Brackets with all other functions equal to zero.

6) The matrix defining the Generalized Poisson Brackets is equal to brackets of functions equal to coordinates:

$$\{f,g\} = \frac{\partial f}{\partial z_k} \cdot \{z_k, z_m\} \cdot \frac{\partial g}{\partial z_m}. \qquad (16)$$

7) The following two equalities hold

$$\{f,g\} = \{f, z_m\} \cdot \frac{\partial g}{\partial z_m}$$
$$and \qquad (17)$$
$$\{f,g\} = \frac{\partial f}{\partial z_k} \cdot \{z_k, g\}.$$

IV. GENERALIZED POISSON BRACKETS ALLOWED BY THE EQUATIONS OF MOTION (DYNAMICALLY ALLOWED BRACKETS)

Infinitely many different Generalized Poisson Bracket structures described in the previous section may be introduced simply by choosing a matrix field $P_{ij}(z,t), \quad i,j=1,...2N$, satisfying conditions (8) - (10), and then using equation (11) to define the Generalized Poisson Brackets. Yet only some of these bracket structures will be compatible with the equations of motion, in the sense that the brackets will reproduce these equations of motion from a function called a Hamiltonian. So we need to study a subset of all Generalized Poisson Brackets that will fit a specific set of the equations of motion.

Assume we have a specific system of the equations of motion (1) and that after the simplification this system takes a form of equations (6) and (7). Generalized Poisson Brackets

{ , } are called Dynamically Allowed Brackets for the equations of motion (1), if they satisfy the compatibility condition:

$$\frac{\partial[\{f,g\}(z,t)]}{\partial z_n}\cdot E_n(z,t)=\{\frac{\partial f(z,t)}{dz_k}\cdot E_k(z,t),g(z,t)\}+\{f(z,t),\frac{\partial g(z,t)}{\partial z_m}\cdot E_m(z,t)\}, \qquad (18)$$

where functions $E_i(z,t)$ are the right sides of the equations (6). Einstein summation notation was used in (18). Notice that the Dynamically Allowed Brackets are not directly affected by constraints (7). Notice also that the condition (18), together with conditions (8) – (10) allow for specifying the Dynamically Allowed Brackets separately at every point of time t, completely independently of how they were defined at other points of time.

Checking the conditions like (18) with arbitrary functions $f(z,t)$ and $g(z,t)$ may be difficult. It is then worth to observe that we need to check much simpler conditions, namely

$$\frac{\partial\{z_i,z_j\}}{\partial z_n}\cdot E_n=\{E_i,z_j\}+\{z_i,E_j\} \qquad i,j=1,...,2N\,, \qquad (19)$$

where z_i's are functions equal to the coordinates. Einstein summation is NOT used on the right side of (19). If we assume (19), then we have

$$\frac{\partial\{f,g\}}{dz_n}\cdot E_n=\frac{\partial}{\partial z_n}\left[\frac{\partial f}{\partial z_k}\cdot\{z_k,z_m\}\cdot\frac{\partial g}{\partial z_m}\right]\cdot E_n=$$

$$=E_n\cdot\frac{\partial}{\partial z_n}\left(\frac{\partial f}{\partial z_k}\right)\cdot\{z_k,z_m\}\cdot\frac{\partial g}{\partial z_m}+\frac{\partial f}{\partial z_k}\cdot\frac{\partial\{z_k,z_m\}}{\partial z_n}\cdot E_n\cdot\frac{\partial g}{\partial z_m}+\frac{\partial f}{\partial z_k}\cdot\{z_k,z_m\}\cdot\frac{\partial}{\partial z_n}\left(\frac{\partial g}{\partial z_m}\right)\cdot E_n=$$

$$=E_n\cdot\frac{\partial}{\partial z_k}\left(\frac{\partial f}{\partial z_n}\right)\cdot\{z_k,z_m\}\cdot\frac{\partial g}{\partial z_m}+\frac{\partial f}{\partial z_k}\cdot\frac{\partial\{z_k,z_m\}}{\partial z_n}\cdot E_n\cdot\frac{\partial g}{\partial z_m}+\frac{\partial f}{\partial z_k}\cdot\{z_k,z_m\}\cdot\frac{\partial}{\partial z_m}\left(\frac{\partial g}{\partial z_n}\right)\cdot E_n=$$

$$=E_n\cdot\frac{\partial}{\partial z_k}\left(\frac{\partial f}{\partial z_n}\right)\cdot\{z_k,z_m\}\cdot\frac{\partial g}{\partial z_m}+\frac{\partial f}{\partial z_k}\cdot\left[\{E_k,z_m\}+\{z_k,E_m\}\right]\cdot\frac{\partial g}{\partial z_m}+$$
$$+\frac{\partial f}{\partial z_k}\cdot\{z_k,z_m\}\cdot\frac{\partial}{\partial z_m}\left(\frac{\partial g}{\partial z_n}\right)\cdot E_n=$$

$$= E_n \cdot \frac{\partial}{\partial z_k}\left(\frac{\partial f}{\partial z_n}\right) \cdot \{z_k, z_m\} \cdot \frac{\partial g}{\partial z_m} + \frac{\partial f}{\partial z_k} \cdot \{E_k, z_m\} \cdot \frac{\partial g}{\partial z_m} + + \frac{\partial f}{\partial z_k} \cdot \{z_k, E_m\} \cdot \frac{\partial g}{\partial z_m} + \frac{\partial f}{\partial z_k} \cdot \{z_k, z_m\} \cdot \frac{\partial}{\partial z_m}\left(\frac{\partial g}{\partial z_n}\right) \cdot E_n =$$

$$= E_n \cdot \{\frac{\partial f}{\partial z_n}, g\} + \frac{\partial f}{\partial z_k} \cdot \{E_k, z_m\} \cdot \frac{\partial g}{\partial z_m} + + \frac{\partial f}{\partial z_k} \cdot \{z_k, E_m\} \cdot \frac{\partial g}{\partial z_m} + \{f, \frac{\partial g}{\partial z_n}\} \cdot E_n =$$

$$= E_k \cdot \{\frac{\partial f}{\partial z_k}, g\} + \frac{\partial f}{\partial z_k} \cdot \{E_k, z_m\} \cdot \frac{\partial g}{\partial z_m} + + \frac{\partial f}{\partial z_n} \cdot \{z_n, E_m\} \cdot \frac{\partial g}{\partial z_m} + \{f, \frac{\partial g}{\partial z_m}\} \cdot E_m =$$

$$= E_k \cdot \{\frac{\partial f}{\partial z_k}, g\} + \frac{\partial f}{\partial z_k} \cdot \{E_k, g\} + \{f, E_m\} \cdot \frac{\partial g}{\partial z_m} + \{f, \frac{\partial g}{\partial z_m}\} \cdot E_m =$$

$$= \{\frac{\partial f}{\partial z_k} \cdot E_k, g\} + \{f, \frac{\partial g}{\partial z_m} \cdot E_m\}.$$

So we got

$$\frac{\partial\{f, g\}}{dz_n} \cdot E_n = \{\frac{\partial f}{\partial z_k} \cdot E_k, g\} + \{f, \frac{\partial g}{\partial z_m} \cdot E_m\},$$

and this is exactly the condition (18).

V. THE DEFINITION AND THE EXISTENCE OF A HAMILTONIAN

Assume we have equations of motion in the form (6), and the Generalized Poisson Brackets allowed for these equations. Define the Hamiltonian function $H(z,t)$ by giving its partial derivatives as:

$$E_i(z,t) = P_{im}(z,t) \cdot \frac{\partial H(z,t)}{\partial z_m}, \tag{20}$$

where $E_i(z,t)$ are the right sides of the equations of motion (6), and the $P_{im}(z,t)$ are the matrices that define the Dynamically Allowed Brackets . Since matrices $P_{im}(z,t)$ are invertible at each (z,t), the equations (20) fully define all the partial derivatives $\frac{\partial H(z,t)}{\partial z_m}$. These derivatives then, for each chosen time, locally define the Hamiltonian $H(z,t)$ up to an additive constant, provided second order mixed derivatives obtained from the definitions (20) are independent of the order of taking the derivatives. So we will check that independence now:

The brackets being Dynamically Allowed Brackets mean $\frac{\partial\{z_i, z_j\}}{\partial z_m} \cdot E_m = \{E_i, z_j\} + \{z_i, E_j\}$.

So we have

$$0 = \frac{\partial\{z_i, z_j\}}{\partial z_m} \cdot E_m + \{z_j, E_i\} + \{E_j, z_i\} =$$

$$= \frac{\partial P_{ij}}{\partial z_m} \cdot E_m + \{z_j, E_i\} + \{E_j, z_i\} = \frac{\partial P_{ij}}{\partial z_m} \cdot P_{mk} \cdot \frac{\partial H}{\partial z_k} + \{z_j, E_i\} + \{E_j, z_i\} =$$

$$= \frac{\partial P_{ij}}{\partial z_m} \cdot P_{mk} \cdot \frac{\partial H}{\partial z_k} + \{z_j, P_{im} \cdot \frac{\partial H}{\partial z_m}\} + \{P_{jm} \cdot \frac{\partial H}{\partial z_m}, z_i\} =$$

$$= \frac{\partial P_{ij}}{\partial z_m} \cdot P_{mk} \cdot \frac{\partial H}{\partial z_k} + P_{im} \cdot \{z_j, \frac{\partial H}{\partial z_m}\} + \{z_j, P_{im}\} \cdot \frac{\partial H}{\partial z_m} + P_{jm} \cdot \{\frac{\partial H}{\partial z_m}, z_i\} + \{P_{jm}, z_i\} \cdot \frac{\partial H}{\partial z_m} =$$

$$= \frac{\partial P_{ij}}{\partial z_m} \cdot P_{mk} \cdot \frac{\partial H}{\partial z_k} +$$

$$+ P_{im} \cdot \frac{\partial z_j}{\partial z_k} \cdot P_{kr} \cdot \frac{\partial^2 H}{\partial z_r \partial z_m} + \frac{\partial z_j}{\partial z_k} \cdot P_{kr} \cdot \frac{\partial P_{im}}{\partial z_r} \cdot \frac{\partial H}{\partial z_m} +$$

$$+ P_{jm} \cdot \frac{\partial^2 H}{\partial z_k \partial z_m} \cdot P_{kr} \cdot \frac{\partial z_i}{\partial z_r} + \frac{\partial P_{jm}}{\partial z_k} \cdot P_{kr} \cdot \frac{\partial z_i}{\partial z_r} \cdot \frac{\partial H}{\partial z_m} =$$

$$= \frac{\partial P_{ij}}{\partial z_m} \cdot P_{mk} \cdot \frac{\partial H}{\partial z_k} +$$

$$+ P_{im} \cdot \delta_{jk} \cdot P_{kr} \cdot \frac{\partial^2 H}{\partial z_r \partial z_m} + \delta_{jk} \cdot P_{kr} \cdot \frac{\partial P_{im}}{\partial z_r} \cdot \frac{\partial H}{\partial z_m} +$$

$$+ P_{jm} \cdot \frac{\partial^2 H}{\partial z_k \partial z_m} \cdot P_{kr} \cdot \delta_{ir} + \frac{\partial P_{jm}}{\partial z_k} \cdot P_{kr} \cdot \delta_{ir} \cdot \frac{\partial H}{\partial z_m} =$$

$$= \frac{\partial P_{ij}}{\partial z_m} \cdot P_{mk} \cdot \frac{\partial H}{\partial z_k} +$$

$$+ P_{im} \cdot P_{jr} \cdot \frac{\partial^2 H}{\partial z_r \partial z_m} + P_{jr} \cdot \frac{\partial P_{im}}{\partial z_r} \cdot \frac{\partial H}{\partial z_m} +$$

$$+ P_{jm} \cdot \frac{\partial^2 H}{\partial z_k \partial z_m} \cdot P_{ki} + \frac{\partial P_{jm}}{\partial z_k} \cdot P_{ki} \cdot \frac{\partial H}{\partial z_m} =$$

$$= \frac{\partial P_{ij}}{\partial z_m} \cdot P_{mk} \cdot \frac{\partial H}{\partial z_k} + P_{jr} \cdot \frac{\partial P_{im}}{\partial z_r} \cdot \frac{\partial H}{\partial z_m} + \frac{\partial P_{jm}}{\partial z_k} \cdot P_{ki} \cdot \frac{\partial H}{\partial z_m} +$$

$$+ P_{im} \cdot P_{jr} \cdot \frac{\partial^2 H}{\partial z_r \partial z_m} + P_{jm} \cdot \frac{\partial^2 H}{\partial z_k \partial z_m} \cdot P_{ki} =$$

$$= \frac{\partial P_{ij}}{\partial z_m} \cdot P_{ms} \cdot \frac{\partial H}{\partial z_s} + P_{jr} \cdot \frac{\partial P_{is}}{\partial z_r} \cdot \frac{\partial H}{\partial z_s} + \frac{\partial P_{js}}{\partial z_k} \cdot P_{ki} \cdot \frac{\partial H}{\partial z_s} +$$

$$+ P_{im} \cdot P_{jr} \cdot \frac{\partial^2 H}{\partial z_r \partial z_m} + P_{ki} \cdot P_{jm} \cdot \frac{\partial^2 H}{\partial z_k \partial z_m} =$$

$$= \left[\frac{\partial P_{ij}}{\partial z_m} \cdot P_{ms} + P_{jr} \cdot \frac{\partial P_{is}}{\partial z_r} + \frac{\partial P_{js}}{\partial z_k} \cdot P_{ki} \right] \cdot \frac{\partial H}{\partial z_s} +$$

$$+ P_{im} \cdot P_{jr} \cdot \frac{\partial^2 H}{\partial z_r \partial z_m} - P_{ik} \cdot P_{jm} \cdot \frac{\partial^2 H}{\partial z_k \partial z_m} =$$

$$= \left[\frac{\partial P_{ij}}{\partial z_m} \cdot P_{ms} + P_{jr} \cdot \frac{\partial P_{is}}{\partial z_r} + \frac{\partial P_{js}}{\partial z_k} \cdot P_{ki} \right] \cdot \frac{\partial H}{\partial z_s} +$$

$$+ P_{im} \cdot P_{jr} \cdot \frac{\partial^2 H}{\partial z_r \partial z_m} - P_{ik} \cdot P_{jr} \cdot \frac{\partial^2 H}{\partial z_k \partial z_r} =$$

$$= \left[\frac{\partial P_{ij}}{\partial z_m} \cdot P_{ms} + P_{jr} \cdot \frac{\partial P_{is}}{\partial z_r} + \frac{\partial P_{js}}{\partial z_k} \cdot P_{ki} \right] \cdot \frac{\partial H}{\partial z_s} +$$

$$+ P_{ik} \cdot P_{jr} \cdot \frac{\partial^2 H}{\partial z_r \partial z_k} - P_{ik} \cdot P_{jr} \cdot \frac{\partial^2 H}{\partial z_k \partial z_r} =$$

$$= \left[\frac{\partial P_{ij}}{\partial z_m} \cdot P_{ms} + P_{jr} \cdot \frac{\partial P_{is}}{\partial z_r} + \frac{\partial P_{js}}{\partial z_k} \cdot P_{ki} \right] \cdot \frac{\partial H}{\partial z_s} +$$

$$+ P_{ik} \cdot P_{jr} \cdot \left[\frac{\partial^2 H}{\partial z_r \partial z_k} - \frac{\partial^2 H}{\partial z_k \partial z_r} \right] =$$

$$= \left[P_{ms} \cdot \frac{\partial P_{ij}}{\partial z_m} + P_{jr} \cdot \frac{\partial P_{is}}{\partial z_r} + P_{ki} \cdot \frac{\partial P_{js}}{\partial z_k} \right] \cdot \frac{\partial H}{\partial z_s} +$$

$$+ P_{ik} \cdot P_{jr} \cdot \left[\frac{\partial^2 H}{\partial z_r \partial z_k} - \frac{\partial^2 H}{\partial z_k \partial z_r} \right] =$$

$$= \left[P_{ms} \cdot \frac{\partial P_{ij}}{\partial z_m} - P_{rj} \cdot \frac{\partial P_{is}}{\partial z_r} + P_{ki} \cdot \frac{\partial P_{js}}{\partial z_k} \right] \cdot \frac{\partial H}{\partial z_s} +$$

$$+ P_{ik} \cdot P_{jr} \cdot \left[\frac{\partial^2 H}{\partial z_r \partial z_k} - \frac{\partial^2 H}{\partial z_k \partial z_r} \right] =$$

$$= \left[P_{ms} \cdot \frac{\partial P_{ij}}{\partial z_m} + P_{rj} \cdot \frac{\partial P_{si}}{\partial z_r} + P_{ki} \cdot \frac{\partial P_{js}}{\partial z_k} \right] \cdot \frac{\partial H}{\partial z_s} +$$

$$+ P_{ik} \cdot P_{jr} \cdot \left[\frac{\partial^2 H}{\partial z_r \partial z_k} - \frac{\partial^2 H}{\partial z_k \partial z_r} \right] =$$

$$= \left[P_{ns} \cdot \frac{\partial P_{ij}}{\partial z_n} + P_{nj} \cdot \frac{\partial P_{si}}{\partial z_n} + P_{ni} \cdot \frac{\partial P_{js}}{\partial z_n} \right] \cdot \frac{\partial H}{\partial z_s} +$$

$$+ P_{ik} \cdot P_{jr} \cdot \left[\frac{\partial^2 H}{\partial z_r \partial z_k} - \frac{\partial^2 H}{\partial z_k \partial z_r} \right] =$$

$$= \left[P_{na} \cdot \frac{\partial P_{ij}}{\partial z_n} + P_{nj} \cdot \frac{\partial P_{ai}}{\partial z_n} + P_{ni} \cdot \frac{\partial P_{ja}}{\partial z_n} \right] \cdot \frac{\partial H}{\partial z_a} +$$

$$+ P_{ik} \cdot P_{jr} \cdot \left[\frac{\partial^2 H}{\partial z_r \partial z_k} - \frac{\partial^2 H}{\partial z_k \partial z_r} \right] =$$

$$= \left[P_{na} \cdot \frac{\partial P_{ij}}{\partial z_n} + P_{ni} \cdot \frac{\partial P_{ja}}{\partial z_n} + P_{nj} \cdot \frac{\partial P_{ai}}{\partial z_n} \right] \cdot \frac{\partial H}{\partial z_a} + P_{ik} \cdot P_{jr} \cdot \left[\frac{\partial^2 H}{\partial z_r \partial z_k} - \frac{\partial^2 H}{\partial z_k \partial z_r} \right] =$$

$$= P_{ik} \cdot P_{jr} \cdot \left[\frac{\partial^2 H}{\partial z_r \partial z_k} - \frac{\partial^2 H}{\partial z_k \partial z_r} \right].$$

We used the condition (10) to get the last step.

So we have

$0 = P_{ik} \cdot P_{jr} \cdot \left[\frac{\partial^2 H}{\partial z_r \partial z_k} - \frac{\partial^2 H}{\partial z_k \partial z_r} \right]$ for any $i, j = 1,...,2N$. Since the matrices P_{nm} are invertible, then for any $k, r = 1,...,2N$ we have

$$\left[\frac{\partial^2 H}{\partial z_r \partial z_k} - \frac{\partial^2 H}{\partial z_k \partial z_r} \right] = 0. \tag{21}$$

Thus, the Hamiltonian $H(z,t)$ is locally well defined, up to a time-dependent additive function.

Now, we will check that the Hamiltonian is working as expected. Let $f(z,t)$ be any function, and assume that the variables (z) in it follow the equations of motion (6). Then we have

$$\frac{df(z,t)}{dt} = \frac{\partial f}{\partial z_k} \cdot \dot{z}_k + \frac{\partial f}{\partial t} = \frac{\partial f}{\partial z_k} \cdot E_k + \frac{\partial f}{\partial t} = \frac{\partial f}{\partial z_k} \cdot P_{km}(z,t) \cdot \frac{\partial H}{\partial z_m} + \frac{\partial f}{\partial t} = \{ f, H \} + \frac{\partial f}{\partial t}.$$

In obtaining the above we used equations (6), (20), and (11). So we have a usual equation for time derivatives of functions:

$$\frac{df(z,t)}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \qquad (22)$$

which justifies the fact that the function defined by equations (20) is called a Hamiltonian.

Applying the equations (22) to the functions equal to coordinates shows that the equations of motions (6) are reproduced by the Dynamically Allowed Brackets and the Hamiltonian as:

$$\dot{z}_i = \{z_i, H\} + \frac{\partial z_i}{\partial t} = \{z_i, H\} = \frac{\partial z_i}{\partial z_k} \cdot P_{km}(z,t) \cdot \frac{\partial H}{\partial z_m} = \delta_{ik} \cdot P_{km}(z,t) \cdot \frac{\partial H}{\partial z_m} = P_{im}(z,t) \cdot \frac{\partial H}{\partial z_m} = E_i$$

In the last step we used the definition (20). So we reproduced the equations (6), $\dot{z}_i = E_i$.

Conversely, assume that we have Generalized Poisson Brackets $\{\ \ ,\ \ \}$ and a function $H(z,t)$ such that $\{z_i, H\} = E_i, \quad i = 1,...,2N$. Then this function is a Hamiltonian, since in an obvious way it reproduces the equations of motion (6)

$$\dot{z}_i = E_i, \quad i = 1,...,2N$$

by the equations

$$\dot{z}_i = \{z_i, H\}, \quad i = 1,...,2N\ .$$

Also, for any other function $f(z,t)$ we obtain

$$\frac{df(z,t)}{dt} = \frac{\partial f}{\partial z_k} \cdot \dot{z}_k + \frac{\partial f}{\partial t} = \frac{\partial f}{\partial z_k} \cdot E_k + \frac{\partial f}{\partial t} = \frac{\partial f}{\partial z_k} \cdot \{z_k, H\} + \frac{\partial f}{\partial t} = \{f, H\} + \frac{\partial f}{\partial t}.$$

(We used property (17) of the Generalized Poisson Brackets in the last step of this calculation.)

So we have:

$$\frac{df(z,t)}{dt} = \{f, H\} + \frac{\partial f}{\partial t}.$$

Finally, this Generalized Poisson Brackets is a bracket that is allowed by the equations of motion (6), since for any $i, j = 1,...,2N$, we have

$$\frac{\partial\{z_i,z_j\}}{\partial z_n}\cdot E_n-\{E_i,z_j\}-\{z_i,E_j\}=$$

$$=\frac{\partial\{z_i,z_j\}}{\partial z_n}\cdot\{z_n,H\}-\{\{z_i,H\},z_j\}-\{z_i,\{z_j,H\}\}=$$

$$=\{\{z_i,z_j\},H\}-\{\{z_i,H\},z_j\}-\{z_i,\{z_j,H\}\}=$$

$$=\{\{z_i,z_j\},H\}+\{\{H,z_i\},z_j\}+\{\{z_j,H\},z_i\}\}=$$

$$=\{\{z_i,z_j\},H\}+\{\{z_j,H\},z_i\}+\{\{H,z_i\},z_j\}=0.$$

The last equation was obtained using the Jacobi identity.

Thus we have

$$\frac{\partial\{z_i,z_j\}}{\partial z_n}\cdot E_n-\{E_i,z_j\}-\{z_i,E_j\}=0,$$

but this is equivalent to equation (19). It means that the Generalized Poisson Brackets, which are producing given equations of motion from a Hamiltonian, automatically are Dynamically Allowed Brackets for these equations.

VI. THE NEED FOR DEFINING THE DYNAMICALLY ALLOWED BRACKETS

The Dynamically Allowed Brackets were defined in section IV as a subset of all Generalized Poisson Brackets, satisfying the conditions (18) or, equivalently, simpler conditions (19). A priori it is possible that all Generalized Poisson Brackets satisfy these conditions for all given systems of equations of motion, and then the separate definition of Dynamically Allowed Brackets would not be needed. To see that the definition of Dynamically Allowed Brackets is indeed needed, consider the following example:

Let the mechanical system have just one spatial dimension, so its position-velocity space is described by the coordinates (x,v). Assume the equations of motion:

$$\dot{x} = v$$

$$\dot{v} = -v - x.$$

Define Generalized Poisson Brackets by $\{x,x\} = \{v,v\} = 0$, $\{x,v\} = 1$, $\{v,x\} = -1$, and then define the brackets for all other functions using equation (16). By direct check, the above satisfies all requirements for Generalized Poisson Brackets. The conditions for the brackets being Dynamically Allowed (19) give

$$\frac{\partial\{x,x\}}{\partial x}\cdot v + \frac{\partial\{x,x\}}{\partial v}\cdot(-v-x) = \{v,x\} + \{x,v\}$$

$$\frac{\partial\{x,v\}}{\partial x}\cdot v + \frac{\partial\{x,v\}}{\partial v}\cdot(-v-x) = \{v,v\} + \{x,-v-x\}$$

$$\frac{\partial\{v,x\}}{\partial x}\cdot v + \frac{\partial\{v,x\}}{\partial v}\cdot(-v-x) = \{-v-x,x\} + \{v,v\}$$

$$\frac{\partial\{v,v\}}{\partial x}\cdot v + \frac{\partial\{v,v\}}{\partial v}\cdot(-v-x) = \{-v-x,v\} + \{v,-v-x\}.$$

The conditions above are not satisfied, since lines 2 and line 3 above produce after simplification $0 = -1$ and $0 = 1$ respectively. So, the Generalized Poisson Brackets defined in this example are not Dynamically Allowed Brackets.

Therefore, for a given set of equations of motion, the Dynamically Allowed Brackets are proper subset of all Generalized Poisson Brackets. The Brackets that are Dynamically Allowed are the Brackets that allow for reproducing the original equations of motion from Hamiltonians, and not all Generalized Poisson Brackets will do that.

VII. THE EXISTENCE OF DYNAMICALLY ALLOWED BRACKETS FOR A GIVEN SET OF EQUATIONS OF MOTION

Assume we have equations of motion in the form (6). Then the "flow box" theorem[4)] tells us that there locally exists, for each time t separately, an invertible change of coordinates

$$w_i = w_i(z), \quad i = 1,...,2N, \tag{23}$$

such that the equations of motion (6), when expressed in the coordinates w_i, become

$$\begin{aligned} &\dot{w}_1 = 1 \\ &\dot{w}_i = 0, \quad i = 2,...,2N. \end{aligned} \tag{24}$$

The same "flow box" theorem may be used for an N-dimensional free particle with unitary mass, described by the coordinates $(x_i, v_i) \quad i = 1,..., N$, with the equations of motion

$$\begin{aligned} &\dot{v}_i = 0 \\ &\dot{x}_i = v_i, \quad i = 1,..., N. \end{aligned} \tag{25}$$

In this case, we have a local invertible change of coordinates

$$w_i = w_i(x, v), \quad i = 1,...,2N \tag{26}$$

such that equations (25) become equations (25) with that change of coordinates.

Combining the two changes of coordinates we obtain a local invertible change of variables

$$z_i = z_i(x_j, v_j), \quad i = 1,...,2N,\ j = 1,..., N, \tag{27}$$

and this change of coordinates will change the equations (6) into equations (25).

The system of equations (25) has a standard set of Poisson Brackets, given by

$$\{x_i, x_j\}_F = \{v_i, v_j\}_F = 0, \qquad \{x_i, v_j\}_F = \delta_{ij}, \tag{28}$$

and a standard Hamiltonian

$$H = \sum_{i=1}^{N} \frac{v_i^2}{2}. \tag{29}$$

Recall that we possibly have a different change of coordinates (27) for every time t , and therefore a different brackets (28) and a different Hamiltonian (29) for each time t. If now, for time t, we define the brackets as

$$\{f(z,t), g(z,t)\} = \{f(z(x,v),t), g(z(x,v),t)\}_F, \tag{30}$$

and the Hamiltonian

$$H = \sum_{i=1}^{N} \frac{v_i^2(z,t)}{2}, \tag{31}$$

then the bracket (30) and the Hamiltonian (31) will reproduce the equations of motion (6), since these are really the same brackets and the same Hamiltonian that produced equations (25), and equations (25) are just equations (6) in different variables. Also, the brackets will satisfy all the conditions placed on the Generalized Poisson Brackets, since they are Generalized Poisson Brackets to start with.

Finally, the brackets (30) will satisfy the conditions for the Dynamically Allowed Brackets (18), because they reproduce the equations of motion from a Hamiltonian and, as shown in the preceding section, they are then automatically Dynamically Allowed Brackets.

Earlier we observed that the Dynamically Allowed Brackets and the Hamiltonian are indeed defined independently for each time t, so if we can do it for one specific time, we can do it for all times, and we obtain the Dynamically Allowed Brackets defined at any time.

So at least one system of Dynamically Allowed Brackets exists for the system of equations (6). Even more, we may observe that, at any one time t, all possible Dynamically Allowed Bracket systems for (6) are obtained from all possible Dynamically Allowed Brackets for rather simple equations (25) of a free particle.

VIII. FINAL REMARKS

Studying the structure of all Allowed Dynamical Brackets of a given set of the equations of motion seem to be quite difficult in the original system of coordinates. It becomes trivial in the "flow box" coordinate system (23) (all Generalized Poisson Brackets are allowed for it, as long as $\{w_i, w_j\}$ are not explicitly dependent on w_1), but then the transformation of these results back to the original system of coordinates may be problematic.

Earlier we described the Dynamically Allowed Brackets at different times t as being not related. This is indeed the case. However, if needed and if the original equations of motion depend in some smooth way on time, then locally we may require similar smoothness from the brackets and the Hamiltonian, at least as long as we deal with finitely dimensional case.

Notice that the constraints (7) play no role in construction we presented. Indeed, they do not – as long as the equations of motion (6) are given also for the points that do not satisfy the constraints, which was our assumption about the initial equations (1). However, if the equations of motion (1) would be given only at the points that satisfy the constraints, and for some reasons it would be impossible to extend these equations to the vicinity of these points in a smooth enough way, the problem would have to be reconsidered.

Finally, please notice that what we presented here may open a path for a direct quantization of the equations of motion. We prove the existence of the Hamiltonian, but in a sense the Hamiltonian is a secondary thought here – the Dynamically Allowed Brackets are obtained directly from the equations of motion. Therefore, even if the Hamiltonian is never calculated, the system may be quantized using the Heisenberg picture. The time evolution of the observable in this picture would be governed directly by the equations of motion rather then the commutator of the observable and the Hamiltonian as is usually accepted.

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