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Generalized Position-Velocity Variables and the Existence of a Lagrange-Hamilton Formalism for Given n-Dimensional Newtonian Equations of Motion

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Given a time-independent system of equations of motion that would not allow the existence of a Lagrangian, generalized position-velocity systems of variables are proposed. It is shown that some of these variable systems will allow the existence of a Lagrangian. It is shown that if a Lagrangian exist, then the Hamiltonian and Poisson Brackets associated with that Lagrangian also exist. A way of constructing an explicit generalized position-velocity system of variables, an explicit Lagrangian, and an associated explicit Hamiltonian and Poisson Brackets is shown for the case when general solutions of the original equations of motion are explicitly known.

I. INTRODUCTION

In our previous work we have shown that a Lagrangian and an associated Hamiltonian with its Poisson Brackets always exist locally for a given system of equations of motion[1]. Also, we explicitly constructed a Lagrangian for a non-holonomic system that traditionally would be considered not to have a Lagrangian[2]. In previous work we were using the variables that were present in the original equations of motion (and some additional extraneous variables as well). In this paper we will show the existence of a Lagrangian by changing from the original variables to more suitable generalized position-velocity variables. The advantage of this approach is a that the calculations become much simpler in comparison to the use of the original variables. For example, there is no need to use the additional extraneous variables. Calculating the Hamiltonian and the Poisson Brackets becomes much simpler as well. The calculations are

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significantly simplified especially, if we want to return to the original variables when calculating the final version of the Hamiltonian and the Poisson Brackets. The same statement cannot be made about the Lagrangian; returning to the original variables in the Lagrangian will not, in general, produce a Lagrangian that would give the correct equations of motion.

The second topic we discuss here is the possibility of getting the generalized position-velocity variables and the Lagrangian explicitly, and therefore also getting an explicit Hamiltonian and explicit Poisson Brackets, in the situation when we explicitly know the solutions of the original equations if motion.

We believe that the methods given here will not only simplify the calculations but will also provide a much better insight to our methods of constructing the Lagrangian than our previous work did.

The organization of our presentation is as follows:

In section II, we will define generalized position-velocity variables, the systems of variables that are potentially suitable for constructing Lagrangians, although not every system of that kind will guarantee the existence of a Lagrangian.

In section III, we recall the so-called box variables, the variables that standardize description of possible motions and their relation to the original variables of the problem.

In section IV, we discuss the existence of a special kind of generalized position-velocity variables, the free variables. We show the existence of the free variables and the relation of these variables to the variables used in earlier sections. Free variables allow for easy construction of a Lagrangian, and consequently also the Hamiltonian and the Poisson Brackets, which are discussed in section V.

Free variables exist for every dynamical system but, in general, we do not know how to construct them explicitly. In a case when the general solutions of the system of equations of motion are explicitly known, the free variables, and consequently the Lagrangian and the rest of the structure, can be, in principle, explicitly constructed. This construction is presented in section VI.

In section VII, the closing remarks, we comment on the relation of this work to some earlier work we did.

II. GENERALIZED POSITION-VELOCITY VARIABLES

Consider a system of particles described locally by n spatial coordinates $(x_1,...,x_n)$. Assume that they satisfy the following system of n second-order, time-independent, equations of motion:

$$\ddot{x}_i = R_i(x_j, \dot{x}_j), \quad i, j = 1,...,n. \tag{1}$$

By introducing velocity variables $(v_1,...,v_n)$, defined by $v_i = \dot{x}_i$, the equations of motion (1) can be rewritten as:

$$\dot{x}_i = v_i \quad i = 1,...,n. \tag{2}$$

$$\dot{v}_i = R_i(x_j, v_j), \quad i, j = 1, ..., n.$$

Now we introduce new $2n$ variables. First define $(y_i), \quad i = 1, ..., n$ as any functions of the previous variables

$$y_i = y_i(x_j, v_j), \quad i, j = 1, ..., n. \tag{3}$$

Then define $(w_i), \quad i = 1, ..., n$ as the time derivatives of the (y_i), with the use of the equations of motion (2). So, for these functions we get

$$w_i = \sum_{k=1}^{n} \left(\frac{\partial y_i}{\partial x_k} v_k + \frac{\partial y_i}{\partial v_k} R_k\right) \quad i = 1, ..., n. \tag{4}$$

In general, for given functions (y_i), the functions (y_i, w_i) may not form a new system of independent variables that could replace the system (x_j, v_j). If they do, we call them generalized position-velocity variables. Also, in such a case the variables (y_i) will be called space-like variables, and the variables (w_i) will be called velocity-like variables.

The main point is that even if the original variables (x_j, v_j) may not allow the existence of a Lagrangian giving the equations (2), the new variables (y_i, w_i) may allow a Lagrangian that will produce the equations (2), although expressed in terms of the variables (y_i, w_i).

Obviously not all new variables will do that, since for example new variables can be identical to the old ones, hence allowing no Lagrangian. But, in the next section we are going to show that locally there always exists a set of generalized position-velocity variables that allow a Lagrangian, no matter what the original equations of motion (2) are.

III. BOX VARIABLES

Let us go back to variables (x_j, v_j), and equations (2). With some restrictions, the box theorem[3)] tells us that instead of the variables (x_j, v_j) we may locally use variables $(z_k) \quad k = 1,...,2n$, such that the equations (2) , when expressed in terms of variables (z_k), become:

$$\dot{z}_1 = 1$$
$$\dot{z}_k = 0, \quad k = 2,...,2n \tag{5}$$

IV. FREE VARIABLES

Let us now introduce another dynamical system described by variables (q_i, u_i) that satisfy the equations:

$$\begin{aligned} \dot{q}_i &= u_i \quad i = 1,...,n. \\ \dot{u}_i &= 0, \quad i = 1,...,n. \end{aligned} \tag{6}$$

We call them free variables.

The box theorem then tells us that instead of the variables (q_i, u_i) we may locally use variables $(z_k) \quad k = 1,...,2n$, that satisfy the equations (5). Since the equations (5) do not change under translations, we can locally match the ranges of the variables (z_k) used in section III with the ranges of the variables (z_k) used here, in section IV. Then we can match the variables as well.

Using the composition of the change of variables from the original variables (x_j, v_j) from section II, to variables (z_k) from section III, and then from (z_k) to (q_i, u_i) from section IV, we get the change of variables from (x_j, v_j) to (q_i, u_i). That change may be expressed as:

$$
\begin{aligned}
q_i &= q_i(x_j, v_j) \\
u_i &= u_i(x_j, v_j),
\end{aligned}
\qquad i, j = 1,...n \tag{7}
$$

or, inverting the change of variables (7), we get

$$
\begin{aligned}
x_i &= x_i(q_j, u_j) \\
v_i &= v_i(q_j, u_j).
\end{aligned}
\qquad i, j = 1,...n \tag{8}
$$

Also, from the way these variables were constructed, we know that the original equations of motion (2), when expressed in variables (q_i, u_i), become the free equations (6). And vice-versa, the free equations (6), when expressed in the variables (x_j, v_j), will become the original equations (2).

In an obvious way, because of how the equations (6) look like, the free variables (7) are generalized position-velocity variables as defined in section II.

V. THE LAGRANGIAN AND THE HAMILTONIAN

Using the variables (q_i, u_i) we can define a Lagrangian as

$$L = \sum_{i=1}^{n} \frac{u_j^2}{2} \qquad (9)$$

This Lagrangian, being a traditional Lagrangian for free system, will obviously give equations of motion (6) as its Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u_i}\right) = \frac{\partial L}{\partial q_i}, \quad i = 1,\dots,n\,. \qquad (10)$$

Then, if we change the variables in (10) from (q_i, u_i) to (x_j, v_j) using (8), we will get the original equations of motion (2). In this sense the Lagrangian (9) is a Lagrangian for the original system (2).

Notice that the change of the variables to (x_j, v_j) must be done after the Euler-Lagrange equations are obtained from the Lagrangian (9) using the variables (q_i, u_i). This is due to the fact that, when deriving the Euler – Lagrange equations, we make assumption that the velocity-like variable is a time derivative of the corresponding space-like variable, and this relation between a space-like variable and its corresponding velocity-like variable is not preserved under the variable change (7) and (8).

Using the Lagrangian (9) and coordinates (q_i, u_i) we obtain the usual Hamiltonian

$$H = \sum_{i=1}^{n} \frac{u_j^2}{2} \qquad (11)$$

We also obtain the usual Poisson Brackets, namely

$\{q_i, q_j\} = 0$

$$\{u_i, u_j\} = 0 \tag{12}$$

$$\{q_i, u_j\} = \delta_{ij}$$

The Hamiltonian (11) may be expressed by the original variables (x_j, v_j), and the Poisson Brackets for the coordinates (x_j, v_j) can defined by using (8) and the usual formulas for the Poisson Brackets, namely

$$\{x_i, x_j\} = \sum_{k=1}^{n} \left(\frac{\partial x_i}{\partial q_k}\frac{\partial x_j}{\partial u_k} - \frac{\partial x_i}{\partial u_k}\frac{\partial x_j}{\partial q_k}\right)$$

$$\{v_i, v_j\} = \sum_{k=1}^{n} \left(\frac{\partial v_i}{\partial q_k}\frac{\partial v_j}{\partial u_k} - \frac{\partial v_i}{\partial u_k}\frac{\partial v_j}{\partial q_k}\right) \tag{13}$$

$$\{x_i, v_j\} = \sum_{k=1}^{n} \left(\frac{\partial x_i}{\partial q_k}\frac{\partial v_j}{\partial u_k} - \frac{\partial x_i}{\partial u_k}\frac{\partial v_j}{\partial q_k}\right)$$

Also, the original equations of motions (2) will be reproduced, as usual, by

$$\dot{x}_i = \{x_i, H\}$$

$$\dot{v}_i = \{v_i, H\} \tag{14}$$

VI. THE CASE OF EXPLICIT SOLUTIONS OF THE EQUATIONS OF MOTION

In some cases, we may have the explicit solutions of the original equations of motion. In such cases it will be possible to explicitly get the variables change (7), (8), and therefore to explicitly perform all the calculations presented above. The outline of the procedure is as follows.

Assume we have explicit general solution of the equations of motion in a $2n$ dimensional position-velocity space. Then locally, around any non-stationary point of that space, we can choose such a $2n-1$ dimensional subset of a $2n$ dimensional open set that the points of the solutions originating from that subset will re-create that local open set.

On this $2n-1$ dimensional subset we can introduce variables $A_i, \quad i=1,...,2n-1$. Locally, using translation we can assure that none of the variables A_i are equal to zero anywhere. Then, using the general solution of the equations of motion with their time parameter t, each point in that local open $2n$ dimensional set may be associated with the parameter t needed to reach that point when starting with t equal zero at a point of the $2n-1$ dimensional subset. Then we use the set of variables $A_i, \quad i=1,...,2n-1$ to describe the originating point in the $2n-1$ dimensional set. Then we can translate t in a such a way that nowhere on the $2n$ dimensional set would t be equal to zero.

The variables $A_i, \quad i=1,...,2n-1$, together with the variable t, become a set of variables on the open $2n$ dimensional set. In these variables the solutions are represented by variables $A_i, \quad i=1,...,2n-1$ being constants of motion, and only the variable t changing with time (actually, equal to time). Since the derivative of time is equal to 1, these variables make one possibility for the box variables described in section III.

Introduce new variables by formulas

$q_1 = A_1 t$

$u_1 = A_1$

$$q_2 = A_2 t + A_3 \qquad (15)$$

$$u_2 = A_2$$

$$q_3 = A_4 t + A_5$$

$$u_3 = A_4$$

…….

$$q_n = A_{2n-2} t + A_{2n-1}$$

$$u_n = A_{2n-2}$$

A direct check gives us the equations:

$$\dot{q}_i = u_i \quad i = 1, \ldots, n$$

$$\dot{u}_i = 0, \quad i = 1, \ldots, n, \qquad (16)$$

which are identical to equations (6). So, the variables (q_i, u_i) are the free variables as described in section IV. Thus, the Lagrangian, the Hamiltonian, and the Poisson Brackets can be defined in terms of these variables, as described in section V.

The procedure described above was successfully used by authors when constructing free variables for a non-holonomic dynamical system[2)], allowing for a construction of a Lagrangian.

VII. FINAL REMARKS

Modern approach to dynamical systems stresses a description that is as coordinate and variable independent as possible. However, in this work we had to use a completely opposite approach, where specific variables were crucial for the results.

In our earlier work a more complicated method of finding a Lagrangian was given. The advantage of that more complicated method was that it used the original variables in the construction of a Lagrangian (as well as some other variables, that were later self-eliminated by the constraints that were showing up in a natural way in the system). The advantage of the method presented here is that it is simpler. The disadvantage is that it is not using the original variables when constructing the Lagrangian, the following Hamiltonian, and the Poisson Brackets. However, at the end of calculations both methods give the same Hamiltonian and the Poisson Brackets, and the Hamiltonian and the Poisson Brackets may be expressed in terms of the original variables. So one may claim that both methods are equivalent.

Finally, we would like to point out that in cases when the procedure outlined here can be done explicitly, we will obtain explicit expressions for canonical variables for the system, since the free variables from section V satisfy the conditions (12) for canonical variables.

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