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Darboux-Box Variables and the Existence of a Complete Set of Lagrangians for Given n-Dimensional Newtonian Equations of Motion. The Isomorphism of n-Dimensional Newtonian Dynamical Systems

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Darboux-Box Variables and the Existence of a Complete Set of Lagrangians for Given n-Dimensional Newtonian Equations of Motion. The Isomorphism of n-Dimensional Newtonian Dynamical Systems

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For given time-independent Newtonian system of equations of motion and given Poisson Brackets allowed by these equations, it is proven that locally a Lagrangian exists that gives these equations of motion as its regular Euler-Lagrange equations, and gives these Poisson Brackets in a regular process of obtaining the Hamiltonian and its Poisson Brackets. However, this Lagrangian may be using generalized position-velocity variables instead of the original position-velocity variables from the original equations of motion. Also, Darboux Box variables are introduced to prove that all Newtonian dynamical systems are locally isomorphic, meaning that locally there exists a one-to-one function relating the variables describing both systems that preserves equations of motion, Poisson Brackets, the Hamiltonian and, in a sense, also the Lagrangian.

I. INTRODUCTION

There exists a common consensus in the available literature that some important Newtonian dynamical systems do not allow the Lagrangian nor the Poisson Brackets, even local ones, that have the usual, essential properties, and therefore they require a very different approach[1)]. We agree with that conclusion only to the extent that in many cases we may not be able to get a Lagrangian and the Poisson Brackets in an explicit form, so in practice we may need to use the other methods. In our previous work we have shown that a Lagrangian and its associated Hamiltonian with its Poisson Brackets always exist locally for a given Newtonian system of equations of motion. In the same work we have described a method of explicitly finding a

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Lagrangian and its Poisson Brackets in cases when explicit solution to equations of motion are known[2]. As an example of the usefulness of this method we explicitly constructed a Lagrangian for a non-holonomic system that traditionally would be considered not to have a Lagrangian nor the Poisson Brackets with the usual properties[3].

In this work we go even further, showing that not only a Lagrangian exists locally for each system of equations of motion, but that a complete set of Lagrangians exists. These Lagrangians may require changing from the original variables to some suitable space-velocity variables, via Darboux-Box variables that we define here; generally different Darboux-Box variables and different space-velocity variables will be required for different Poisson Brackets structures. This set of Lagrangians is complete in the sense that as a set they will generate all possible Hamiltonians with all possible systems of Poisson Brackets allowed for the given equations of motion, by the usual process of getting a Hamiltonian and its Poisson Brackets. Consequently, some Lagrangians obtained by methods shown in this work, while in general different from traditional Lagrangians, will generate the same Hamiltonian and Poisson Brackets as traditional Lagrangians do, (as well as some different ones). So, all traditional Hamiltonian – Poisson Brackets systems are obtained from these new Lagrangians, but this approach also covers the cases when a traditional Lagrangian does not exist. Since all possible Hamiltonian systems are covered, one may claim that possibly all physical systems associated with given equations of motion are also covered, all having their Lagrangians, and therefore their canonical systems of Poisson Brackets.

It seems that these different Lagrangians will, in general, produce systems with different physical properties. At this point we have no way to distinguish which possibility will actually realize itself in the real life, but we also know that some systems that are physically very different may in fact have the same equations of motion. Maybe this variety of structures reflects the possible variety of different physical systems sharing the same equations of motion.

There is a second way of looking at our result though. Instead of interpreting it as a mere change of coordinates into more suitable for creating a Lagrangian for a given system, one may interpret it as an isomorphism between any two, possibly different, Newtonian dynamical systems of the same dimension. This isomorphism is then preserving the Hamiltonian and the Poisson Bracket structure between the systems, while in general not preserving the Lagrangian. Still, one may claim that in a sense both systems are "identical" in that both can be presented in variables for which the Lagrangian exists, or they may not, while the relation between these variables and common Darboux-Box coordinates will be identical in both cases. So maybe the specific variables in which each system was originally formulated is not so important and with this interpretation the systems become fully isomorphic (indistinguishable).

The organization of our presentation is as follows:

In section II, we will describe the Newtonian dynamical systems we consider in this work and recall the definition of Poisson Brackets allowed by a given dynamical system.

In section III, we show that for a given dynamical system and given Poisson Brackets allowed by that coordinate system a Darboux-Box coordinate system always exists. In these coordinates the equations of motion become a box-type, while at the same time the coordinates are the canonical coordinates for the given Poisson Brackets.

In section IV, we show that Darboux-Box coordinates from section III can be transformed to free space-velocity variables in such a way that these space-velocity variables are still canonical variables with respect to the Poisson Brackets from section III. We also show that the usual Lagrangian for the free variables becomes the Lagrangian of the dynamical system from section II, the usual Poisson Brackets obtained from this Lagrangian become the Poisson Brackets from section II, and the usual Hamiltonian obtained from that Lagrangian becomes the Hamiltonian of the dynamical system from the section II.

In section V, we briefly describe the possibility of replacing the Lagrangian in section VI by a Lagrangian that uses the original variables of the dynamical system instead of the free space-velocity variables used in section IV.

In section VI, the closing remarks, we point out that the free space-velocity system used in section III to obtain a specific Lagrangian is not the only system that can be used, and that obtaining a Lagrangian does not have to be the only goal. More general, identifying Darboux-Box variables for any two systems is a way to show an isomorphism between two systems of the same dimension.

II. THE NEWTONIAN DYNAMICAL SYSTEM

Consider a given system of particles described locally by n spatial coordinates $(x_1,...,x_n)$.

Assume that they satisfy the following system of n second-order, time-independent, equations of motion:

$$\ddot{x}_i = R_i(x_j, \dot{x}_j), \quad i, j = 1,...,n. \tag{1}$$

By introducing velocity variables $(v_1,...,v_n)$, defined by $v_i = \dot{x}_i$, the equations of motion (1) can be rewritten as:

$$\begin{aligned} \dot{x}_i &= v_i \quad i = 1,...,n. \\ \dot{v}_i &= R_i(x_j, v_j), \quad i, j = 1,...,n. \end{aligned} \tag{2}$$

At this point it is convenient to change the notation, just for the remainder of this section. We will introduce $\alpha_i = x_i \;\; i = 1,...,n$, and $\alpha_i = v_{i-n} \;\; i = n+1,...,2n$. If convenient, we will use (α) when referring to these coordinates. The equations (2) can then be written in the form:

$$\dot{\alpha}_i = E_i(\alpha) \qquad i = 1,...,2n \tag{3}$$

Let us start with recalling a definition[4)] of dynamically allowed Poisson Brackets for the system (3). The Poisson Brackets $\{\ ,\ \}$ are called dynamically allowed by the equations (2) (or equivalently (1) or (3)) if they satisfy the condition

$$\sum_{k=1}^{2n} \frac{\partial\{\alpha_i, \alpha_j\}}{\partial \alpha_k} \cdot E_k = \{E_i, \alpha_j\} + \{\alpha_i, E_j\} \qquad i, j = 1,...,2n \tag{4}$$

It can be shown[4)] that condition (4) is equivalent to the existence of a Hamiltonian that will reproduce the equations (2) by the usual Hamilton equations.

$$\dot{x}_i = \{x_i, H\}, \quad i = 1,\ldots,n,$$
$$\dot{v}_i = \{v_i, H\}, \quad i = 1,\ldots,n. \tag{5}$$

where the Poisson Brackets described before are used.

In other words, the Hamiltonian and the Poisson Brackets satisfy the equations

$$\{x_i, H\} = v_i \quad i = 1,\ldots,n,$$
$$\{v_i, H\} = R_i(x_j, v_j), \quad i, j = 1,\ldots,n. \tag{6}$$

where $R_i(x_j, v_j), \quad i, j = 1,\ldots,n$ are taken from equations (2).

The equivalence between the condition (4) and the existence of a Hamiltonian is of “if and only if” type, meaning that if the Hamiltonian giving equations (6) exists, then the Poisson Brackets used in (6) will satisfy the condition (4). Vice versa, if given Poisson Brackets satisfy the condition (4), then a Hamiltonian giving (6) will exists[4)].

As a result of the equations (5) and (6), for any dynamical variable $f = f(x_i, v_i) \quad i = 1,\ldots,n$, we also have

$$\dot{f} = \{f, H\}. \tag{7}$$

In our previous work[5)] we have shown that a Lagrangian (and therefore also a Hamiltonian with its Poisson Brackets) locally exists for any system of equations of motion. This shows that the set of the Poisson Brackets that are dynamically allowed by a given system of equations of motion is not empty.

In the following sections we want to show that for a given system of equations of motion, and a given system of Poisson Brackets that are dynamically allowed by these equations of motion, there exists a Lagrangian that will reproduce these equations of motion as its Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial u_i}\right)=\frac{\partial L}{\partial q_i}, \quad i=1,...,n. \tag{8}$$

However, it should be noted that the variables used in the equations (8) are not, in general, the original position-velocity variables used in the original equations of motion (2), but rather the generalized position-velocity variables[5)].

III. THE DARBOUX-BOX VARIABLES

Let us start with a given system of equations of motion (2), and a given system of Poisson Brackets dynamically allowed by these equations, denoted by $\{\ ,\ \}$. Since the brackets $\{\ ,\ \}$ are dynamically allowed, then a Hamiltonian H also exists. Then the Box Theorem[6)] tells us that locally there exists a system of variables $(z_i)\quad i=1,...,2n$

$$z_i=z_i(x_j,v_j), \quad i=1,...,2n, \quad j=1,...,n, \tag{9}$$

such that the equations of motion (2), when expressed by the variables (9), become

$$\dot{z}_1 = 1$$
$$\dot{z}_i = 0, \quad i = 2,...,2n. \tag{10}$$

If we express the Hamiltonian and the Poisson Brackets in variables (9), we get

$$\dot{z}_1 = \{z_1, H\}$$
$$\dot{z}_i = \{z_i, H\}, \quad i = 2,...,2n. \tag{11}$$

Therefore, we also get

$$\{z_1, H\} = 1 \tag{12}$$

$$\{z_i, H\} = 0, \quad i = 2,...,2n. \tag{13}$$

The structure of the Poisson Brackets is[4)]

$$\{f, g\} = \sum_{i,j=1}^{2n} \frac{\partial f}{\partial z_i} \cdot P_{ij} \cdot \frac{\partial g}{\partial z_j}, \tag{14}$$

where P_{ij} is an antisymmetric matrix. Then using (12) and (14) we have

$$1 = \{z_1, H\} = \sum_{i,j=1}^{2n} \delta_{1i} \cdot P_{ij} \cdot \frac{\partial H}{\partial z_j} = \sum_{i=1}^{2n} P_{1j} \cdot \frac{\partial H}{\partial z_j} = \sum_{i=2}^{2n} P_{1j} \cdot \frac{\partial H}{\partial z_j} = 1. \tag{15}$$

The last step was possible since P_{ij} equals to zero, since it is an antisymmetric matrix. Using the last equality in (15), we get

$$\sum_{i=2}^{2n} P_{1j} \cdot \frac{\partial H}{\partial z_j} = 1. \tag{16}$$

This means that at least one partial derivative of the Hamiltonian H with respect to $z_i, \quad i \neq 1$

must be non-zero. Without loss of generality, let us say that $\frac{\partial H}{\partial z_2} \neq 0.$ This means that locally

the variables $(z_1, H, z_3, z_4, ..., z_{2n})$ can replace the variables $(z_1, z_2, z_3, z_4, ..., z_{2n})$. Since the time derivative of the Hamiltonian H is zero, the variables $(z_1, H, z_3, z_4, ..., z_{2n})$ are also box variables. Equations of motions (2) expressed by these variables become:

$$\begin{aligned} &\dot{z}_1 = 1 \\ &\dot{H} = 0 \\ &\dot{z}_i = 0, \quad i = 3, ..., 2n. \end{aligned} \tag{17}$$

Now, on a temporary basis, let us introduce a completely new dynamic system on the variables $z_i, \quad i = 3, ..., 2n.$ This new temporary dynamic system has nothing to do with the dynamic system given by the original equations of motion (2). It is used only to introduce new, convenient variables. The parameter derivatives of this new system are given by

$$\mathring{z}_i, = \{z_i, z_1\} \quad i = 3, ..., 2n, \tag{18}$$

where the small circle over the variables represents a derivative with respect to the parameter associated with that temporary dynamic. The Poisson Brackets in (18) are still the “old” Poisson Brackets specified at the beginning of the section III.

Then the Box Theorem[6)] used again tells us that there exist such variables

$$y_i = y_i(z_j), \quad i, j = 3, ..., 2n, \tag{19}$$

that

$$\overset{\circ}{y}_3 = 1, \tag{20}$$

$$\overset{\circ}{y}_i = 0, \quad i = 4,\ldots,2n. \tag{21}$$

Combining (20) and (21) with (18) we get

$$\{y_3, y_1\} = 1, \tag{22}$$

$$\{y_i, y_1\} = 0, \quad i = 4,\ldots 2n. \tag{23}$$

At this point we stop using the temporary dynamic. It was only needed to define the variables (19) and the results (22) and (23).

Now let us make another change of variables. Define

$$Y_1 = z_1 \tag{24}$$

$$Y_2 = H \tag{25}$$

$$Y_3 = y_3 - H \tag{26}$$

$$Y_i = y_i, \quad i = 4,\ldots,2n. \tag{27}$$

A direct check shows that

$$\{Y_1, Y_2\} = 1, \tag{28}$$

$$\{Y_1, Y_3\} = 0, \tag{29}$$

$$\{Y_1, Y_i\} = 0, \quad i = 4,\ldots,2n, \tag{30}$$

$$\{Y_2, Y_3\} = 0, \tag{31}$$

$$\{Y_2, Y_i\} = 0, \quad i = 4,...,2n. \tag{32}$$

Let us now look at the variables $Y_i, \quad i = 3,...,2n.$ By using the Darboux Theorem[7)] we can replace them by new variables

$$\beta_i = \beta_i(Y_j), \quad i = 2,...,n, \quad j = 3,...,2n \tag{33}$$

$$\gamma_i = \gamma_i(Y_j), \quad i = 2,...,n, \quad j = 3,...,2n \tag{34}$$

such that the Poisson Brackets of these new variables are

$$\{\beta_i, \beta_j\} = 0, \quad i, j = 2,...,n \tag{35}$$

$$\{\gamma_i, \gamma_j\} = 0, \quad i, j = 2,...,n \tag{36}$$

$$\{\beta_i, \gamma_j\} = \delta_{ij}, \quad i, j = 2,...,n \tag{37}$$

where $\delta_{ij}, \quad i, j = 3,...,n$ is a Kronecker Delta. Let us stress that the Poisson Brackets in equations (35), (36), and (37) are still the same Poisson Brackets that we started the section III with. They have not changed, only the variables changed.

Now define

$$\beta_1 = Y_1, \tag{38}$$

$$\gamma_1 = Y_2. \tag{39}$$

With this definition we have the variables $(\beta_i, \gamma_i), \quad i = 1,...,n,$ in which the original equations of motion (2) can be written as:

$$\dot{\beta}_1 = 1, \tag{40}$$

$$\dot{\beta}_i = 0, \quad i = 2,\ldots,n, \tag{41}$$

$$\dot{\gamma}_i = 0, \quad i = 1,\ldots,n. \tag{42}$$

The Poisson Brackets for these coordinates are given as

$$\{\beta_i, \beta_j\} = 0, \quad i, j = 1,\ldots,n \tag{43}$$

$$\{\gamma_i, \gamma_j\} = 0, \quad i, j = 1,\ldots,n \tag{44}$$

$$\{\beta_i, \gamma_j\} = \delta_{ij}, \quad i, j = 1,\ldots,n \tag{45}$$

The variables $(\beta_i, \gamma_i), \quad i = 1,\ldots,n,$ can be called the Darboux-Box Variables.

The Darboux-Box coordinates are invariant under translation. In other words, an arbitrary number can be added separately to any coordinate, and the equations of motion and the Poisson Brackets will still look like equations (40) – (45).

IV. THE LAGRANGIAN OF A FREE SYSTEM AND THE DARBOUX-BOX VARIABLES

Consider a system of free particles described locally by n spatial coordinates $(q_1,\ldots,q_n)$. Particles being free mean that they satisfy the following system of n second-order, time-independent, equations of motion:

$$\ddot{q}_i = 0, \quad i = 1,\ldots,n. \tag{46}$$

By introducing velocity variables $(u_1,\ldots,u_n)$, defined by $u_i = \dot{q}_i$, the equations of motion (46) can be rewritten as:

$$\begin{aligned} \dot{q}_i &= u_i \\ \dot{u}_i &= 0 \end{aligned} \qquad i = 1,\ldots,n. \tag{47}$$

The usual Lagrangian for the system of equations (47) may be defined as:

$$L = \sum_{k=1}^{n} \frac{u_k^2}{2}. \tag{48}$$

It is easy to check the Euler-Lagrange equations (8) of the Lagrangian (48) will match the free equations (46).

The usual Hamiltonian obtained from the Lagrangian (48) is

$$H = \sum_{k=1}^{n} \frac{u_k^2}{2}. \tag{49}$$

The Hamiltonian (49) together with the usual Poisson brackets defined as:

$$\begin{aligned} \{q_k, q_l\} &= 0 \\ \{u_k, u_l\} &= 0 \\ \{q_k, u_l\} &= \delta_{kl}, \end{aligned} \qquad \text{where} \qquad k, l = 0,\ldots n. \tag{50}$$

reproduces the equations of motion by:

$$\begin{aligned} \dot{q}_i &= \{q_i, H\} \\ \dot{u}_i &= \{u_i, H\} \end{aligned} \qquad i = 1,\ldots,n. \tag{51}$$

Then the same process of obtaining the Darboux-Box variables described in section III may be performed here, resulting in the variables $(\beta_i, \gamma_i), \quad i = 1, ..., n,$ in which the original equations of motion (47) can be written as:

$$\dot{\beta}_1 = 1, \tag{52}$$

$$\dot{\beta}_i = 0, \quad i = 2, ..., n, \tag{53}$$

$$\dot{\gamma}_i = 0, \quad i = 1, ..., n. \tag{54}$$

The Poisson Brackets for these coordinates will be given as

$$\{\beta_i, \beta_j\} = 0, \quad i, j = 1, ..., n \tag{55}$$

$$\{\gamma_i, \gamma_j\} = 0, \quad i, j = 1, ..., n \tag{56}$$

$$\{\beta_i, \gamma_j\} = \delta_{ij}, \quad i, j = 1, ..., n \tag{57}$$

Then the variables $(\beta_i, \gamma_i), \quad i = 1, ..., n,$ from this section may be locally identified with the variables $(\beta_i, \gamma_i), \quad i = 1, ..., n,$ from section III (the ranges may be adjusted by translation that is allowed for these variables). This will then produce a change of variables

$$q_i = q_i(x_j, v_j), \quad i, j = 1, ..., n, \tag{58}$$

$$u_i = u_i(x_j, v_j), \quad i, j = 1, ..., n, \tag{59}$$

with the inverse

$$x_i = x_i(q_j, u_j), \quad i, j = 1, ..., n, \tag{60}$$

$$v_i = v_i(q_j, u_j), \quad i, j = 1, ..., n. \tag{61}$$

The original equations of motion (2), when expressed in variables $(q_i, u_i), \quad i = 1, ..., n,$ will become free equations of motion (47). Vice-versa, free equations of motion (47), when expressed in variables $(x_i, v_i), \quad i = 1, ..., n,$ will become the original equations (2). The Poisson Brackets (55), (56), and (57), when expressed in variables $(x_i, v_i), \quad i = 1, ..., n,$ will become the original Poisson Brackets from section II, and vice-versa, the Poisson Brackets from section II, when expressed in coordinates in coordinates $(q_i, u_i), \quad i = 1, ..., n,$ will become the Poisson Brackets (55), (56), (57). The Hamiltonian in both coordinate systems is the same, we just recalculate it from one system of coordinates to the other.

In this sense we can claim that the Lagrangian (48) is re-creating the original equations (2) and the original General Poisson Brackets from section II.

We need to stress that when calculating the Euler-Lagrange equations we must, in general, obtain the equations using the variables $(q_i, u_i), \quad i = 1, ..., n.$ This is due to the fact that the Euler-Lagrange equations explicitly identify each velocity with its originating coordinate, and this identification is not, in general, preserved under the coordinate change (60) and (61).

V. LAGRANGIAN WITH ORIGINAL VARIABLES

It is possible to use a Lagrangian that will utilize the original variables $(x_i, v_i), \quad i = 1,...,n,$ instead of the variables $(q_i, u_i), \quad i = 1,...,n.$ The procedure, called Spontaneous Dimension Reduction[2], starts with a Lagrangian that uses the original variables as well as some additional variables. The Lagrangian will be made of a free Lagrangian, as well as parts similar to constraints imposing the change of variables (58) – (61), together with variables of the type of the Lagrangian Multipliers. All extraneous variables will then be spontaneously removed from the system by constraints that are obtained from the Lagrangian as some of the Euler-Lagrange equations. If the Dirac procedure for constraints[8),9)] is then used, the resulting Hamiltonian and the Poisson Brackets will be identical to these obtained in the current work.

VI. CLOSING REMARKS

In section IV, we used a free particle system to create a specific Lagrangian and the rest of the dynamical structure. However, creating the specific Lagrangian may not be the main goal here. Much more important may be the observation that by identifying Darboux-Box variables of any two dynamical systems of the same dimension we obtain an isomorphism of these two systems. This isomorphism will preserve the equations of motion, the Poisson Brackets and the Hamiltonian, while not preserving the Lagrangian.

We do not have to identify just two systems. We can identify all of them. Some of them will have the Lagrangians, some will not. We can then look at all the Lagrangians not as specific Lagrangians for specific cases, but as all Lagrangian allowed in different variables for essentially just one existing dynamical system. Some variables will allow a Lagrangian, others will not but, in a sense, locally it is the same one system.

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