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An explicit construction of mechanically correct Lagrangians for systems with linear nonholonomic constraints

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An explicit construction of mechanically correct Lagrangians for systems with linear nonholonomic constraints

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Starting with an unconstrained mechanical system that is governed by an initial unconstrained Lagrangian, subsequently modified by nonholonomic, linear in velocities, constraints, an explicit construction for a Lagrangian that will produce mechanically correct equations of motion for that constrained nonholonomic system is given. Obtaining a Hamiltonian from that Lagrangian is briefly discussed.

I. INTRODUCTION

While the Newtonian equations of motion seem to be physically more fundamental than the Lagrangian that produces these equations as its Euler-Lagrange equations, the Lagrangian is of great interest, since it provides a natural framework for further studying of the system. For example, it is a starting point for calculating the Hamiltonian and the Poisson brackets structure, with the quantization possibly following. The problem of constructing a Lagrangian and the following Hamiltonian for given equations of motion has been therefore extensively studied, but it is still not completely resolved. [1)]

Quite often we want to study a mechanical system for which a Lagrangian is already known, but which is subsequently modified by imposition of additional constraints. The constraints usually modify the original equations of motion, and the modifications then lead to the need of modifications of the Lagrangian. Modifying the Lagrangian is quite simple, if the constraints are

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of holonomic type (constraints that could be expressed by restricting the allowable positions of the system). In this case the new Lagrangian is obtained by adding the constraints, each constraint multiplied by its own so called Lagrange multiplier, treated as a new independent variable, to the original Lagrangian.[2)] In the case of nonholonomic constraints (these are constraints that involve velocities and cannot be reduced to restricting the positions only) the situation is not so simple. Adding these constraints multiplied by the Lagrange multipliers to the original Lagrangian will produce equations of motion that are correct from the mathematical point of view, but they are different from real-life mechanical equations that result from such constraints. Specifically, in the case of nonholonomic constraints, the constraints forces resulting from the use of Lagrange multipliers do not satisfy the condition of zero virtual work, which is expected to be satisfied in real-world mechanics. The Lagrange multipliers approach gives us so called vakonomic systems, instead of the real-life mechanical systems. [3)]

In the case of a real-life mechanical system, a commonly accepted approach for nonholonomic case is not to modify the Lagrangian at all, but to obtain the Euler-Lagrange equations from the original Lagrangian, and then modify these equations to include external forces resulting from the constraints.[3)] Consequently, using this approach when obtaining a Hamiltonian system from that Lagrangian, we first obtain the Hamiltonian with its equations of motion and Poisson Brackets, and only then the equations are modified, similarly to how we modified the Euler-Lagrange equations. However, since the resulting equations of motion are not the usual Euler-Lagrange and Hamiltonian equations, many advantages of using that structure are lost. For example, a quantization process must somehow include the direct quantization of the equations of motion, instead of the familiar canonical quantization of the Hamiltonian and the Poisson Brackets.

In this work we attempt to modify the Lagrangian to obtain the correct equations of motion. Starting with the existing Lagrangian, we extend the system by introducing additional, non-physical, spatial-type coordinates, together with a number of “virtual” holonomic constraints on the spatial coordinates of that extended variable system. These “virtual” holonomic constraints are explicitly related to the actual nonholonomic constraints we want to impose. The “virtuality” of these constraints means that, while they place restrictions on the possible spatial variables of the extended variable system, they do not impose any restrictions on the original variables. Still, these holonomic constraints when added to the initial Lagrangian by the means of Lagrangian multipliers, modify the equations of motions involving the original variables to the form required by the real-life mechanics.

At this stage, some of the variables of the system will not have time derivatives determined by the Euler-Lagrange equations. So some gauge-type freedom is presented in the system. This freedom is then removed by imposing the original nonholonomic constraints together with some additional holonomic constraints involving original and added spatial variables. We impose these additional constraints without modifying the existing equations of motion in any way, therefore the modification of the Lagrangian is also not needed. Imposing these constraints produce the final effect of getting mechanically proper equations of motion, that satisfy the nonholonomic constraints.

To achieve the above effect, the number of additional variables that we introduce must be infinite. However, we avoid the problem with the interpretation of these additional variables, because all the additional variables are automatically removed from the system by the Euler-Lagrange equations of the modified Lagrangian and the imposed holonomic and nonholonomic constraints. These equations give all new variables as explicit functions of the original variables, making them redundant and therefore ignorable.

The organization of our presentation is as follows:

In section II, we recall the mechanically correct equations of motion for the nonholonomic system.

In section III, we present the proposed Lagrangian and the additional variables we introduce.

In section IV, we calculate the Euler-Lagrange equations for that Lagrangian, we impose additional constraints, and we show that the system of equations obtained this way is identical to the mechanically correct system from section II, and that the additional variables that we introduced are redundant.

In section V, we comment on a possibility to use Dirac's Theory of Constraints[4,5)] to obtain the Hamiltonian formalism for our Lagrangian.

II. MECHANICALLY CORRECT EQUATIONS FOR TO SYSTEMS WITH LINEAR NONHOLONOMIC CONSTRAINTS

A priori, the constraints may modify the existing equations of motion of a system in an arbitrary way. The modification that is realized in real life mechanical systems, based on the assumption of zero virtual work, is described in many sources[3)]. Following them, let us say we have a system with no constraints for which the equations of motion are obtained from the Lagrangian:

$$L_I = L_I(q,\dot{q}). \tag{1}$$

Assume n is the spatial dimension of that system, and $q=(q_1,...,q_n)$ are the generalized coordinates of that system. The subscript I next to L stands for "initial."

Then assume that m nonholonomic and /or holonomic constraints in the form

$$\sum_{j=1}^{n} a_{k,j}(q_1,...,q_n)\cdot\dot{q}_j = 0, \qquad k = 1,...,m, \qquad m \le n, \tag{2}$$

are imposed on that system. We assume $a_{k,j}(q_1,...,q_n)$ to be smooth enough functions of q. Then, if this is a real-life mechanical system (meaning the system satisfying the zero virtual work principle), the equations of motion are:

$$\begin{aligned} &\frac{d}{dt}\left[\frac{\partial L_I}{\partial \dot{q}_i}\right] - \frac{\partial L_I}{\partial q_i} = \sum_{k=1}^{m} \lambda_k \cdot a_{k,i}(q_1,...,q_n), \qquad i = 1,...,n, \\ &\sum_{j=1}^{n} a_{k,j}(q_1,...,q_n)\cdot\dot{q}_j = 0, \qquad k = 1,...,m, \qquad m \le n, \end{aligned} \tag{3}$$

where $\lambda_k = \lambda_k(q,\dot{q})$ will be obtained from the condition that solutions of (3) must obey constraints (2), which are now included among the equations of motion (3).

We should make two observations:

1) None of the equations in (3) are the usual Euler-Lagrange equations obtained from the Lagrangian (1).

2) It can be checked directly that a Lagrangian that is often proposed as the Lagrangian to be used for nonholonomic system, namely

$$L = L(q,\dot{q},\lambda,\dot{\lambda}) = L_I(q,\dot{q}) + \sum_{k=1}^{m} \lambda_k \cdot \left[\sum_{j=1}^{n} a_{k,j}(q_1,...,q_n)\cdot\dot{q}_j\right], \tag{4}$$

will not produce the equations (3) as its Euler-Lagrange equations.

III. THE PROPOSED LAGRANGIAN

Now, assume the same unconstrained system as before, with the Lagrangian (1), on which constraints (2) are imposed. Consider the following Lagrangian:

$$
\begin{aligned}
&L = L(q,\dot{q},w,\dot{w},z,\dot{z},\lambda,\dot{\lambda},\mu,\dot{\mu}) = \\
&= L_I(q,\dot{q}) + \sum_{k=1}^{m} \lambda_k \cdot \left[\sum_{j=1}^{n} a_{k,j}(w_1,...,w_n) \cdot q_j - z_k \right] + \\
&+ \sum_{k=1}^{m} \lambda_{1,k} \cdot \left(z_{1,k} - z_k \right) + \sum_{k=1}^{m} \lambda_{2,k} \cdot \left(z_{2,k} - z_{1,k} \right) + \sum_{k=1}^{m} \lambda_{3,k} \cdot \left(z_{3,k} - z_{2,k} \right) + \sum_{k=1}^{m} \lambda_{4,k} \cdot \left(z_{4,k} - z_{3,k} \right) .... + \\
&+ \sum_{j=1}^{n} \mu_{1,j} \cdot \left(w_{1,j} - w_j \right) + \sum_{j=1}^{n} \mu_{2,j} \cdot \left(w_{2,j} - w_{1,j} \right) + \sum_{j=1}^{n} \mu_{3,j} \cdot \left(w_{3,j} - w_{2,j} \right) + \sum_{j=1}^{n} \mu_{4,j} \cdot \left(w_{4,j} - w_{3,j} \right) + ... \quad .
\end{aligned}
\tag{5}
$$

To create this Lagrangian we introduced infinitely many new non-physical variables

$$
w_j, z_k, \lambda_k, w_{l,j}, z_{l,k}, \lambda_{l,k}, \mu_{l,j}, \qquad j=1,...,n, \quad k=1,...,m, \quad l=1,...,\infty. \tag{6}
$$

These variables are treated on equal footing with other variables.

IV. THE EULER-LAGRANGE EQUATIONS OF MOTION

In general, the Euler-Lagrange equations for the Lagrangian (5) are

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right]=\frac{\partial L}{\partial q_i}, \qquad i=1,\ldots,n,\\
&\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{w}_i}\right]=\frac{\partial L}{\partial w_i}, \qquad i=1,\ldots,n,\\
&\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{z}_k}\right]=\frac{\partial L}{\partial z_k}, \qquad k=1,\ldots,m,\\
&\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\lambda}_k}\right]=\frac{\partial L}{\partial \lambda_k}, \qquad k=1,\ldots,m,\\
&\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{w}_{l,i}}\right]=\frac{\partial L}{\partial w_{l,i}}, \qquad i=1,\ldots,n, \quad l=1,\ldots,\infty,\\
&\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{z}_{l,k}}\right]=\frac{\partial L}{\partial z_{l,k}}, \qquad k=1,\ldots,m, \quad l=1,\ldots,\infty,\\
&\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\lambda}_{l,k}}\right]=\frac{\partial L}{\partial \lambda_{l,k}}, \qquad k=1,\ldots,m, \quad l=1,\ldots,\infty,\\
&\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\mu}_{l,i}}\right]=\frac{\partial L}{\partial \mu_{l,i}}, \qquad i=1,\ldots,n, \quad l=1,\ldots,\infty.
\end{aligned} \tag{7}
$$

Performing the specific calculations in (7) for the Lagrangian (5), we get the following Euler-Lagrange equations:

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{\partial L_I}{\partial \dot{q}_i}\right]-\frac{\partial L_I}{\partial q_i}=\sum_{k=1}^{m}\lambda_k\cdot a_{k,i}(w_1,\dots,w_n), \qquad i=1,\dots,n,\\
&0=\sum_{j=1}^{n}\frac{\partial a_{k,j}(w_1,\dots,w_n)}{\partial w_i}\cdot q_j-\mu_{1,i}, \qquad i=1,\dots,n,\\
&0=-\lambda_k-\lambda_{1,k}, \qquad k=1,\dots,m,\\
&0=\sum_{j=1}^{n}a_{k,j}(w_1,\dots,w_n)\cdot q_j-z_k, \qquad k=1,\dots,m,\\
&0=\mu_{l,i}-\mu_{l+1,i}, \qquad i=1,\dots,n, \quad l=1,\dots,\infty,\\
&0=\lambda_{l,k}-\lambda_{l+1,k}, \qquad k=1,\dots,m, \quad l=1,\dots,\infty,\\
&0=z_{1,k}-z_k, \qquad k=1,\dots,m,\\
&0=z_{l+1,k}-z_{l,k}, \qquad k=1,\dots,m, \quad l=1,\dots,\infty,\\
&0=w_{1,i}-w_i, \qquad i=1,\dots,n,\\
&0=w_{l+1,i}-w_{l,i}, \qquad i=1,\dots,n, \quad l=1,\dots,\infty.
\end{aligned}
\tag{8}
$$

Now we impose more constraints on that system. The first set of constraints are the nonholonomic constraints (2) that we want to impose on the existing system (8), namely

$$\sum_{j=1}^{n}a_{k,j}(q_1,\dots,q_n)\cdot\dot{q}_j=0, \qquad k=1,\dots,m, \qquad m\le n. \tag{9}$$

We also impose additional holonomic constraints

$$w_i-q_i=0, \qquad i=1,\dots,n. \tag{10}$$

As usual, the way the constraints modify the equations of motion and the Lagrangian is, from the mathematical point of view, arbitrary. From the physical point of view, we want to get the mechanically correct equations (3). Therefore, we may decide that these constraints will not modify the equations of motion and the Lagrangian at all. Consequently, we just add the equations (9) and (10) to the system (8), getting:

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{\partial L_I}{\partial \dot{q}_i}\right]-\frac{\partial L_I}{\partial q_i}=\sum_{k=1}^{m}\lambda_k\cdot a_{k,i}(w_1,\ldots,w_n), \qquad i=1,\ldots,n,\\
&\sum_{j=1}^{n}a_{k,j}(q_1,\ldots,q_n)\cdot\dot{q}_j=0, \qquad k=1,\ldots,m, \qquad m\le n,\\
&w_i-q_i=0, \qquad i=1,\ldots,n,\\
&0=\sum_{j=1}^{n}\frac{\partial a_{k,j}(w_1,\ldots,w_n)}{\partial w_i}\cdot q_j-\mu_{1,i}, \qquad i=1,\ldots,n,\\
&0=-\lambda_k-\lambda_{1,k}, \qquad k=1,\ldots,m,\\
&0=\sum_{j=1}^{n}a_{k,j}(w_1,\ldots,w_n)\cdot q_j-z_k, \qquad k=1,\ldots,m,\\
&0=\mu_{l,i}-\mu_{l+1,i}, \qquad i=1,\ldots,n, \quad l=1,\ldots,\infty,\\
&0=\lambda_{l,k}-\lambda_{l+1,k}, \qquad k=1,\ldots,m, \quad l=1,\ldots,\infty,\\
&0=z_{1,k}-z_k, \qquad k=1,\ldots,m,\\
&0=z_{l+1,k}-z_{l,k}, \qquad k=1,\ldots,m, \quad l=1,\ldots,\infty,\\
&0=w_{1,i}-w_i, \qquad i=1,\ldots,n,\\
&0=w_{l+1,i}-w_{l,i}, \qquad i=1,\ldots,n, \quad l=1,\ldots,\infty.
\end{aligned}
\tag{11}
$$

Using simple manipulation, the equations (11) can be simplified to:

$$
\begin{aligned}
&\frac{d}{dt}\left[\frac{\partial L_I}{\partial \dot{q}_i}\right]-\frac{\partial L_I}{\partial q_i}=\sum_{k=1}^{m}\lambda_k\cdot a_{k,i}(q_1,\dots,q_n), \qquad i=1,\dots,n,\\
&\sum_{j=1}^{n}a_{k,j}(q_1,\dots,q_n)\cdot\dot{q}_j=0, \qquad k=1,\dots,m, \qquad m\le n,\\
&w_i=q_i, \qquad i=1,\dots,n,\\
&z_k=\sum_{j=1}^{n}a_{k,j}(q_1,\dots,q_n)\cdot q_j, \qquad k=1,\dots,m,\\
&\mu_{l,i}=\sum_{j=1}^{n}\frac{\partial a_{k,j}(q_1,\dots,q_n)}{\partial q_i}\cdot q_j, \qquad i=1,\dots,n, \quad l=1,\dots,\infty,\\
&\lambda_{l,k}=-\lambda_k, \qquad k=1,\dots,m, \quad l=1,\dots,\infty,\\
&z_{l,k}=\sum_{j=1}^{n}a_{k,j}(q_1,\dots,q_n)\cdot q_j, \qquad k=1,\dots,m, \quad l=1,\dots,\infty,\\
&w_{l,i}=q_i, \qquad i=1,\dots,n, \quad l=1,\dots,\infty.
\end{aligned}
\tag{12}
$$

Now, arguing exactly like in the case of the standard approach, we may conclude that the first two equations in (12) will lead, without solving the equations of motion, to the consistency conditions in the form

$$\lambda_k=\lambda_k\left(q,\dot{q}\right) \tag{13}$$

Using (13) in (12) gives us the final form of the equations of the system, namely:

$$\begin{aligned} &\frac{d}{dt}\left[\frac{\partial L_I}{\partial \dot{q}_i}\right] - \frac{\partial L_I}{\partial q_i} = \sum_{k=1}^{m} \lambda_k \cdot a_{k,i}(q_1,...,q_n), \qquad i=1,...,n, \\ &\sum_{j=1}^{n} a_{k,j}(q_1,...,q_n)\cdot \dot{q}_j = 0, \qquad k=1,...,m, \qquad m \le n, \end{aligned} \tag{14}$$

together with

$$\begin{aligned} &w_i = q_i, \qquad i=1,...,n, \\ &z_k = \sum_{j=1}^{n} a_{k,j}(q_1,...,q_n)\cdot q_j, \qquad k=1,...,m, \\ &\mu_{l,i} = \sum_{j=1}^{n} \frac{\partial a_{k,j}(q_1,...,q_n)}{\partial q_i}\cdot q_j, \qquad i=1,...,n, \quad l=1,...,\infty, \\ &\lambda_{l,k} = -\lambda_k\left(q,\dot{q}\right), \qquad k=1,...,m, \quad l=1,...,\infty, \\ &z_{l,k} = \sum_{j=1}^{n} a_{k,j}(q_1,...,q_n)\cdot q_j, \qquad k=1,...,m, \quad l=1,...,\infty, \\ &w_{l,i} = q_i, \qquad i=1,...,n, \quad l=1,...,\infty. \end{aligned} \tag{15}$$

Looking at equations (14) and (15) we can conclude that equations (14) are identical to the expected equations (3) of the real-life mechanical nonholonomic system, while the equations (15) show all extra variables that we introduced as functions of the initial variables $\left(q,\dot{q}\right)$. Therefore, these additional variables are simply redundant variables on the space described by the original variables, and therefore they may be ignored.

V. CONCLUDING COMMENTS

The approach we are showing in this work should be more suitable for the quantization than the traditional approach, since the equations of motion are obtained in complete form from the Lagrangian, provided we include the constraints (9) and (10) into the system. This is not the case in the standard approach, where the basic equations of motion (3) are not obtained from the Lagrangian, even when constraints are added to the system.

Since our Lagrangian (5) is degenerate to the extreme, with only some velocities expressible by the canonical momenta, the Dirac's Theory of Constraints[4,5)] is a natural choice for creating the Hamiltonian formalism. It seems that obtaining a Hamiltonian and the Dirac's Brackets will be straightforward, and that the Hamiltonian will directly produce the equations of motion, with no further modification needed, except imposing additional constraints.

In our approach the Hamiltonian is likely to give the equations of motion directly from the Hamiltonian. Then some of the constraints may be eliminated by the Dirac's Brackets[4,5)], and the constraints do not get eliminated may be imposed as conditions on the wave function used in this quantization.

REFERENCES

[1]R. M. Santilli, *Foundations of Theoretical Mechanics* (Springer-Verlag, New York, 1978) vol. 1.

[2]H. Goldstein, C. P. Poole and J. L. Safko, *Classical Mechanics, 3rd Ed.* (Addison-Wesley, Boston, 2002) p. 46

[3]A. M. Bloch, *Nonholonomic Mechanics and Control* (Springer, New York, 2003) p.12

[4]P. A. M. Dirac, Can. J. Math. **2**, 129 (1950)

[5]P. A. M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science, New York, 1964)