A Minimal Completion of Doubly Stochastic Matrix

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Table of contents

1 Introduction

2 Main Results

3 Future Research
The term stochastic matrix goes back at least to Romanovsky (1931,p.267). In 1935, Romanovsky provided a detail discussion of stochastic matrices such as the use of it in the theory of discrete markov chains. Doubly stochastics matrices are also called Schur transformations, or they are said to be bistochastic. The term doubly stochastic apperas to have been first introduced by Feller. Stochastics matrices arise in the study of Markov chains and in the variety of modeling problems such as economics and operation research.
An $n \times n$ matrix $A = (a_{ij})$ is called doubly stochastic if

$$a_{ij} \geq 0$$

for all $i, j$, and

$$\sum_{i=1}^{n} a_{ij} = 1$$

for all $j$, and

$$\sum_{j=1}^{n} a_{ij} = 1$$

for all $i$. 
A matrix $B = (b_{ij})$ is called doubly substochastic if

$$b_{ij} \geq 0$$

for all $i, j$,

$$\sum_{i=1}^{n} b_{ij} \leq 1$$

for all $j$, and

$$\sum_{j=1}^{n} b_{ij} \leq 1$$

for all $i$. 
A matrix $P \in M_n(\mathbb{R})$ is said to be a partial permutation matrix if it has at most one nonzero entry in each row and column, and these nonzero entries are all one.
**Theorem (Birkhoff’s Theorem)**

A matrix $A \in M_n(\mathbb{R})$ is a doubly stochastic matrix if and only if for some $N < \infty$ there are permutation matrices $P_1, \cdots, P_2 \in M_n(\mathbb{R})$ and positive scalars $\alpha_1, \cdots, \alpha_N \in \mathbb{R}$ such that $\alpha_1 + \cdots + \alpha_N = 1$ and $A = \alpha_1 P_1 + \cdots + \alpha_N P_N$. 
Graphs and Matching

Definition

A **bipartite graph** $G = (U, V, E)$ is a graph whose vertices can be divided into two disjoint sets such that no edge connects two vertices of the same set.

Definition

A **perfect matching** is a subset of the edge set $E$ such that every vertex has exactly one edge incident on it, i.e, $|U| = |V| = n$.
Perfect Matching

The proof of Birkhoff’s theorem uses the following key Lemma.

**Lemma**

*The associated graph of any doubly stochastic matrix has a perfect matching.*
Perfect Matching

We represent each row and each column in our doubly stochastic matrix with a vertex and we connect the vertex representing row $i$ with the vertex representing row $j$ if the entry $x_{ij}$ in the matrix is not zero. For instance, let\[ A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} & \frac{5}{12} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{12} \end{bmatrix} \]be a doubly stochastic matrix. Then the graph associated to matrix $A$ is given in the picture below.
Proof of Birkhoff’s Theorem

We proceed by induction on the number of nonzero entries in the matrix. Let $M_0$ be a doubly stochastic matrix. By the previous lemma, the associated graph has a perfect matching. For example, in the associated graph previously $(1, 3), (2, 1), (3, 2)$ is a perfect matching and we underline $x_{13}, x_{21}$ and $x_{32}$. Thus we underline exactly one element in each row and each column. Let $\alpha_0$ be the minimum of the underlined entries. Let $P_0$ be the permutation matrix that has a 1 exactly at the position of the underlined elements. If $\alpha_0 = 1$ then all underlined entries are 1, and $M_0 = P_0$ is a permutation matrix.
If $\alpha_0 < 1$ then the matrix $M_0 - \alpha_0 P_0$ has non-negative entries, and the sum of the entries in any row or any column is $(1 - \alpha_0)$. Dividing each entry by $(1 - \alpha_0)$ in $M_0 - \alpha_0 P_0$ gives a doubly stochastic matrix $M_1$. Thus we may write $M_0 = \alpha_0 P_0 + (1 - \alpha_0)M_1$ where $M_1$ is not only doubly stochastic, but has less non-zero entries than $M_0$. 
Proof of Birkhoff’s Theorem

By our induction hypothesis $M_1$ may be written as

$$M_1 = \alpha_1 P_1 + \cdots + \alpha_n P_n$$

where $P_1, \cdots, P_n$ are permutation matrices, and $\alpha_1 P_1 + \cdots + \alpha_n P_n$ is a convex combination. But then we have

$$M_0 = \alpha_0 P_0 + (1 - \alpha_0) \alpha_1 P_1 + \cdots + (1 - \alpha_0) \alpha_n P_n$$

where $P_0, P_1, \cdots, P_n$ are permutation matrices,
Proof of Birkhoff’s Theorem

Thus we have a convex combination since, \(\alpha_0 \geq 0\), each \((1 - \alpha_0)\alpha_i\) is nonnegative and we have

\[
\alpha_0 + (1 - \alpha_0)(\alpha_1 + \cdots + \alpha_n) = \alpha_0 + (1 - \alpha_0) = 1.
\]
**Example of Birkhoff’s Theorem**

A doubly stochastic matrix is a matrix where the entries are non-negative real numbers and the sum of the entries in each row and each column is 1. Let

\[
A = \begin{bmatrix}
\frac{7}{12} & 0 & \frac{5}{12} \\
\frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\
\frac{1}{4} & \frac{1}{2} & \frac{1}{4}
\end{bmatrix}
\]

be a doubly stochastic matrix. Let

\[
P_0 = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

and \(\alpha_0 = \frac{1}{6}\). Thus we get

\[
M_1 = \frac{1}{\frac{1}{6}} (M_0 - \frac{1}{6} P_0)
\]

Then

\[
M_1 = \begin{bmatrix}
\frac{7}{10} & 0 & \frac{3}{10} \\
0 & \frac{3}{5} & \frac{2}{5} \\
\frac{3}{10} & \frac{2}{5} & \frac{3}{10}
\end{bmatrix}
\]
Example of Birkhoff’s Theorem

The graph associated to $M_1$ is the following.

1

2

3

1

2

3
Example of Birkhoff’s Theorem

A perfect matching corresponding to the graph of $M_1$ is

$\{(1,1), (2,2), (3,3)\}$, the associated matrix is $P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

and we have $\alpha_1 = \frac{3}{10}$. Thus we get $M_2 = \frac{1}{1-\frac{3}{10}}(M_1 - \frac{3}{10}P_1)$. Thus

$$M_2 = \begin{bmatrix} 4/7 & 0 & 3/7 \\ 0 & 3/7 & 4/7 \\ 3/7 & 4/7 & 0 \end{bmatrix}.$$
Example of Birkhoff’s Theorem

The graph associated to $M_2$ is the following.

1
2
3

1
2
3
Example of Birkhoff’s Theorem

A perfect matching in this graph is \( \{(1, 3), (2, 2), (3, 1)\} \), the associated permutation matrix is \( P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \), and we have \( \alpha_2 = \frac{3}{7} \). Thus we get \( M_3 = \frac{1}{1-\frac{3}{7}} (M_2 - \frac{3}{7}P_2) \). This gives us that

\[
M_3 = P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.
\]
Example of Birkhoff’s Theorem

Example

Since \( M_2 = \alpha_2 P_2 + (1 - \alpha_2) M_3 = \frac{3}{7} P_2 + \frac{4}{7} P_3 \) and

\[
M_1 = \alpha_1 P_1 + (1 - \alpha_1) M_2 = \frac{3}{10} P_1 + \frac{7}{10} M_2 = \frac{3}{10} P_1 + \frac{3}{10} P_2 + \frac{4}{10} P_3
\]

then

\[
M_0 = \alpha_0 P_0 + (1 - \alpha_0) M_1 = \frac{1}{6} P_0 + \frac{1}{4} P_1 + \frac{1}{4} P_2 + \frac{1}{3} P_3.
\]

thus by Birkhoff’s Theorem a doubly stochastic matrix \( M_0 \) can be written as a convex combination of permutation matrices.
Theorem

Let $B = (b_{ij})_{i,j=1}^n$ be an $n \times n$ doubly substochastic matrix. Denote the sum of all elements of $B$ by $s$, i.e,

$$s = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij}.$$ 

Then the sub-defect of $B$ is $k = \lceil n - s \rceil$, where $\lceil x \rceil$ is the ceiling of $x$, that is the smallest integer greater than or equal to $x$. 
Then the following is true:

(a) there exits an \((n + k) \times (n + k)\) doubly stochastic matrix containing \(B\) as a submatrix;

(b) there does not exist an \((n + l) \times (n + l)\) doubly stochastic matrix containing \(B\) as a submatrix with \(l < k\).
Proof.

Let $r_i$ and $c_i$ be the sum of the $i$th row and $i$th column respectively, namely

$$r_i = \sum_{j=1}^{n} b_{ij} , \quad c_i = \sum_{j=1}^{n} b_{ji}$$

for $i = 1, 2, \ldots, n$. Thus since B is substochastic, then we have

$$0 \leq r_i \leq 1 , \quad 0 \leq c_i \leq 1. \quad (1)$$
Proof.

Induction on sub-defect $k$.

Base case: Let

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$$

be an $n \times n$ doubly substochastic matrix with sub-defect $k = 1$. □
Proof of the main result

Proof.

Then

\[
\begin{pmatrix}
  b_{11} & b_{12} & \ldots & b_{1n} & 1 - r_1 \\
  b_{21} & b_{22} & \ldots & b_{2n} & 1 - r_2 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  b_{n1} & b_{n2} & \ldots & b_{nn} & 1 - r_n \\
  1 - c_1 & 1 - c_2 & \ldots & 1 - c_n & 1 - (n - s)
\end{pmatrix}
\]

is an \((n + 1) \times (n + 1)\) doubly stochastic matrix containing \(B\) as the \(n \times n\) principal submatrix.
Proof of the main result

Proof.

On one hand, according to (1)

\[ 0 \leq 1 - r_i \leq 1, \quad 0 \leq 1 - c_i \leq 1 \]

for \( i = 1, 2, \ldots, n \), and \( 0 \leq n - s \leq 1 \) due to the assumption \( k = \lceil n - s \rceil = 1 \). Hence all elements are between 0 and 1.

\[ \square \]
Proof.

On the other hand,

\[
\sum_{i=1}^{n} (1 - r_i) = n - \sum_{i=1}^{n} r_i = n - s
\]

and

\[
\sum_{i=1}^{n} (1 - c_i) = n - \sum_{i=1}^{n} c_i = n - s.
\]

Hence, each row and column has sum 1.
Proof.

Induction assumption: Suppose $\forall n$ and $k = \lceil n - s \rceil = t \geq 2$, there exists an $(n + t) \times (n + t)$ doubly stochastic matrix such that $B$ is the $n \times n$ principal submatrix. Inductive step: We need to show

$\forall n$, if the sub-defect $k = t + 1$, then there exists an $(n + t + 1) \times (n + t + 1)$ doubly stochastic matrix such that $B$ is the $n \times n$ principal submatrix.
Proof of the main result

Proof.

First build a matrix $\tilde{B}$ as the following

$$\tilde{B} = \begin{pmatrix}
  b_{11} & b_{12} & \cdots & b_{1n} & x_1 \\
  b_{21} & b_{22} & \cdots & b_{2n} & x_2 \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  b_{n1} & b_{n2} & \cdots & b_{nn} & x_n \\
  y_1 & y_2 & \cdots & y_n & 0
\end{pmatrix}$$
Proof of the main result

Proof.

*Note $x_i, y_i$ are

\[
0 \leq x_i \leq 1 - r_i, \quad 0 \leq y_i \leq 1 - c_i
\]

and

\[
\sum_{i=1}^{n} x_i = 1, \quad \sum_{i=1}^{n} y_i = 1
\]

for $i = 1, 2, \ldots, n.$
Proof.

The existence of such $x_i$ and $y_i$ is implied by $k - 1 < \sum_{i=1}^{n}(1 - r_i) = \sum_{i=1}^{n}(1 - c_i) = n - s \leq k$. One such choice is

$$x_1 = 1 - r_1, \quad y_1 = 1 - c_1$$

and

$$x_i = \min\{1 - r_i, 1 - \sum_{j=1}^{i-1} x_j\}, \quad y_i = \min\{1 - c_i, 1 - \sum_{j=1}^{i-1} y_j\}$$

for $i = 2, 3, \ldots, n$. \qed
Proof of the main result

Proof.

Note that $\tilde{B}$ is an $(n + 1) \times (n + 1)$ doubly substochastic matrix with the sum of all elements $s + 2$ and hence the sub-defect $k = \lceil (n + 1) - (s + 2) \rceil = \lceil n - s - 1 \rceil = t$. Due to the assumption, there exists an $[(n + 1) + t] \times [(n + 1) + t]$ doubly stochastic matrix containing $\tilde{B}$ as the $(n + 1) \times (n + 1)$ principal submatrix meaning that $B$ is the $n \times n$ principle submatrix.
Lemma

An $n \times n$ doubly substochastic matrix $B$ can be dilated to a $2n \times 2n$ doubly stochastic matrix.
Proof.

Let $B \in M_n(\mathbb{R})$ be doubly substochastic, let $e = [1, \cdots, 1]^T \in \mathbb{R}^n$, and let $D_r = \text{diag}(Be)$ and $D_c = \text{diag}(B^T e)$ be diagonal matrices containing the row and the column sums of $B$. The doubly stochastic $2n \times 2n$ matrix

$$
\begin{bmatrix}
B & I - D_r \\
I - D_c & B^T
\end{bmatrix}
$$

is a dilation of $B$. Conversely, it is evident that any square submatrix of a doubly stochastic matrix is doubly substochastic.
Let $B$ be a given $n \times n$ doubly substochastic matrix.

1. Complete $B$ to $A$ by using previous lemma.

2. Use Birkhoff’s Theorem and write

$$A = c_1 P_1 + c_2 P_2 + \ldots + c_m P_m$$

as convex combination of permutation matrices, where $A$ and $P_1, \ldots, P_m$ are $2n \times 2n$ matrices, $\sum_{i=1}^{\infty} c_i = 1$ and $c_i \geq 0$, $N < \infty$. Next, write

$$B = c_1 Q_1 + \ldots + c_m Q_m$$

where $Q_i$ is the $n \times n$ principal submatrix of $P_i$ for $i = 1, \ldots, m$, and $Q_i$ are partial permutation matrices.
Complete each partial permutation matrix $Q_i$ to a permutation matrix $\tilde{Q}_i$.

The minimum size of a completion of $B$ is the difference between the size of given doubly substochastic matrix and the number of minimum 1’s that $Q_i$ contains.
Example

Let

\[ B = \begin{pmatrix}
0.2010 & 0.1360 & 0.0030 \\
0.2210 & 0.5420 & 0.2340 \\
0.3440 & 0.3190 & 0.2010 \\
\end{pmatrix} \]

be the $3 \times 3$ principal submatrix of the doubly stochastic matrix. In order to find minimal completion of $B$, we follow the procedure given previously. Use previous lemma to complete $B$ to $A$ as it is shown on the next slide.
Using Birkoff’s Theorem, we write $A$ as a convex combination of partial permutation matrices.
Future Research

Example

\[ A = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix} + 0.2210 \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} + 0.2010 \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} + 0.1360 \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} + 0.0030 \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix} \]
### Example

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
+ 0.3410
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
+ 0.0980
\]
Next, we extract $B$, the 3x3 principle submatrix

<table>
<thead>
<tr>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B =$</td>
</tr>
<tr>
<td>$0.2210 \begin{pmatrix} 0 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix} + 0.2010 \begin{pmatrix} 1 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \end{pmatrix} + 0.1360 \begin{pmatrix} 0 &amp; 1 &amp; 0 \ 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 0 \end{pmatrix} +$</td>
</tr>
<tr>
<td>$0.0030 \begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 0 &amp; 0 \ 1 &amp; 0 &amp; 0 \end{pmatrix} + 0.3410 \begin{pmatrix} 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \ 1 &amp; 0 &amp; 0 \end{pmatrix} + 0.0980 \begin{pmatrix} 0 &amp; 0 &amp; 1 \ 0 &amp; 0 &amp; 0 \ 0 &amp; 1 &amp; 0 \end{pmatrix}$</td>
</tr>
</tbody>
</table>

We then complete each 3x3 partial permutation matrix by adding one row and one column.
## Future Research

### Example

$$\tilde{B} = 0.2210 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + 0.2010 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} +$$

$$0.1360 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} + 0.0030 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} +$$

$$0.3410 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + 0.0980 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} +$$
Thus we have $\hat{B}$, which is the doubly stochastic completion of $B$, with $B$ as 3x3 principle submatrix.

Example

$$
\hat{B} = 
\begin{pmatrix}
0.2010 & 0.1360 & 0.0030 & 0.6600 \\
0.2210 & 0.5420 & 0.2340 & 0.0030 \\
0.3440 & 0.3190 & 0.2010 & 0.1360 \\
0.2340 & 0.0030 & 0.5620 & 0.2010
\end{pmatrix}
$$
Thank you