Classical Solutions of the Generalized Camassa-Holm Equation

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Abstract

In this poster, well-posedness in $C^1(\mathbb{R})$ (a.k.a. classical solutions) for a generalized Camassa-Holm equation (g-kbCH) having $(k+1)$-degree nonlinearities is explored. This result holds for the Camassa-Holm, the Degasperi-Procesi and the Novikov equations, which improves upon earlier results in Sobolev and Besov spaces.

Main Ingredients

For $k \in \mathbb{Z}^+$ and $b \in \mathbb{R}$, we consider the Cauchy problem for the following generalized Camassa-Holm (g-kbCH) equation

$$\begin{cases} (1-\partial_x^2)\partial_t u = u^k \partial_x^3 u + b u^{k-1}\partial_x u \partial_x^2 u - (b+1)u^k \partial_x u, \\ u(x,0) = u(0), \quad x \in \mathbb{R} \text{ and } t \in \mathbb{R}, \end{cases}$$

which takes the following non-local form

$$\partial_t u + u^k \partial_x u + F(u) = 0,$$

$$F(u) = \partial_x (1-\partial_x^2)^{-1}\left[\frac{b}{k+1}u^{k+1} + \frac{3k-b}{2}u^{k-1}(\partial_x u)^2\right] + (1-\partial_x^2)^{-1}\left[\frac{(k-1)(b-k)}{2}u^{k-2}(\partial_x u)^3\right].$$

Theorem 1 *(Picard-Lindelöf) Let X be a Banach space. Suppose that $f : X \to X$ is locally Lipschitz on a closed ball $\bar{B}_R(u_0) \subset X$ where $R > 0$ and $u_0 \in X$. Let*

$$M = \sup_{u \in \bar{B}_R(u_0)} \|f(u)\| < \infty.$$

Then the initial value problem

$$\begin{cases} \dot{u} = f(u) \\ u(0) = u_0 \end{cases}$$

has a continuously differentiable local solution $u(t)$. This solution is defined in the time interval $t \in (-\delta, \delta)$ where $\delta = R/M$.

Objective

We will show that this family of shallow water wave equations is well-posed in the space of bounded and continuously differentiable functions on the real line, denoted C^1, and equipped with the norm

$$\|f\|_{C^1} = \sup_{x\in\mathbb{R}} |f(x)| + \sup_{x\in\mathbb{R}} |\frac{d}{dx} f(x)|.$$

Methodology

We begin by showing how one formally constructs an equivelent ODE system to the g-kbCH equation. Assuming a solution, u, exists and is a C^1 solution of the g-kbCH initial value problem, we have our trajectories satisfy the ODE

$$\begin{cases} \eta_t(x,t) = u^k(\eta(x,t),t) \\ \eta(x,0) = x. \end{cases}$$

Moreover, the above ODE has a unique solution $\eta(x,t)$ which is also continuously differentiable, therefore, we may define

$$w(x,t) = u(\eta(x,t),t), \quad v(x,t) = u_x(\eta(x,t),t), \quad q(x,t) = \eta_x(x,t), \tag{3}$$

and we see that we may easily obtain $u(x,t)$ from the composition

$$u = w \circ \eta^{-1}.$$

We will first find a system of equations satisfied by w, v and q, and then show that this system of equations is indeed an ODE system, and therefore the solutions are uniquely defined. Using w, we will then construct η and u similarly to the above formal definitions.

The Semi-linear System

$$\begin{cases} \frac{d}{dt} w = -P_1(w,v,q) - R_1(w,v,q), \\ \frac{d}{dt} v = \frac{k-b}{2} w^{k-1} v^2 + \frac{b}{k+1} w^{k+1} - P_2(w,v,q) - R_2(w,v,q), \\ \frac{d}{dt} q = k w^{k-1} v q, \end{cases} \tag{1}$$

with initial data

$$\begin{cases} w(x,0) = u_0(x) \\ v(x,0) = \frac{d}{dx} u_0(x) \\ q(x,0) = 1, \end{cases} \tag{2}$$

where

$$P_1(w,v,q) \doteq \frac{1}{2}\int_x^\infty e^{-|\int_x^z q(y,t)dy|}\left(\frac{b}{k+1}w^{k+1}q + \frac{3k-b}{2}w^{k-1}v^2 q\right)(z,t)dz - \frac{1}{2}\int_{-\infty}^x e^{-|\int_x^z q(y,t)dy|}\left(\frac{b}{k+1}w^{k+1}q + \frac{3k-b}{2}w^{k-1}v^2 q\right)(z,t)dz.$$

$$P_2(w,v,q) = \frac{1}{2}\int e^{-|\int_x^z q(y,t)dy|}\left[\frac{b}{k+1}w^{k+1} + \frac{3k-b}{2}w^{k-1}v^2\right] q dz$$

$$R_1(w,v,q) \doteq \frac{(k-1)(b-k)}{4}\int e^{-|\int_x^z q(y,t)dy|} w^{k-2}(z,t)v^3(z,t)q(z,t)dz.$$

$$R_2(w,v,q) \doteq \frac{(k-1)(b-k)}{2}\frac{1}{2}\int_x^\infty e^{-|\int_x^z q(y,t)dy|} w^{k-2}(z,t)v^3(z,t)q(z,t)dz - \frac{(k-1)(b-k)}{2}\frac{1}{2}\int_{-\infty}^x e^{-|\int_x^z q(y,t)dy|} w^{k-2}(z,t)v^3(z,t)q(z,t)dz.$$

Summary of Proof for Well-posedness

- We show that the forcing terms from the o.d.e. (1) are locally Lipschitz in the space $C^1 \times C \times C$.
- This gives us a unique solution (w,v,q) within the time interval $t \in [-T,T]$ where $T = \min\left\{\frac{1}{2\|u_0\|_{C^1}^k}, \frac{1}{2L}\right\}$, where L is our Lipschitz constant.
- We construct our solution $u(x,t)$ from our particle trajectories $\eta : \mathbb{R} \to \mathbb{R}$ and show uniqueness and continuous dependence on the initial data.

References

- J. Holmes, R. Thompson, *Classical Solutions for the Generalized Camassa-Holm Equation*, Submitted to Advances in Differential Equations, (2015).
- A. Himonas, R. Thompson. *Persistence properties and unique continuation for a generalized Camassa-Holm equation*, Journal of Mathematical Physics, **55** 091503 (2014).

Future Research

- Classical solutions to the CH2 system and other shallow water wave equations.
- Asymptotic profiles and propagation speed of solutions to other shallow water wave systems.
- Local and global solutions to Camassa-Holm type equations in Besov spaces.

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