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For a Given, Time Independent, System of Equations of Motion, Any Non-Trivial Constant of Motion is Locally a Hamiltonian

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For a Given, Time Independent, System of Equations of Motion, Any Non-Trivial Constant of Motion is Locally a Hamiltonian

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Given time-independent system of equations of motion and given a local, non-trivial constant of motion for these equations, it is shown that there exists a locally defined system of dynamical brackets, such that this given constant of motion, used together with these dynamical brackets, becomes a local Hamiltonian for the given system of equations of motion.

I. INTRODUCTION

Usually, a Hamiltonian of a mechanical system is obtained by starting with a Lagrangian for that system and performing the Legendre transformation[1]. A different process is also possible, in which we start with the equations of motion of a system, then we define a set of Generalized Poisson Brackets suitable for these equations of motion, and finally we define a Hamiltonian.[2] Here we will show third possible way of obtaining a Hamiltonian. We will show that if we already know a non-trivial constant of motion for a given system of equations of motion, then there exists a set of Generalized Poisson Brackets,[3] such that this constant of motion, when used together with this system of brackets, will reproduce the given equations of motion. In other words, we will show that any non-trivial constant of motion, when used with a suitable Generalized Poisson Brackets, becomes a Hamiltonian.

The organization of our presentation is as follows:

In section II, we will specify the form of the equations of motion that we will be considering, and

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we define a non-trivial constant of motion.

In section III, we show that for a given set of equations of motion and a non-trivial constant of motion, there exists a set of Generalized Poisson Brackets such that this constant of motion becomes a Hamiltonian that reproduces the original equations of motion when using these Generalized Poisson Brackets.

II. EQUATIONS OF MOTION AND A NON-TRIVIAL CONSTANT OF MOTION

Consider a system described locally by n spatial coordinates $(x_1,...,x_n)$. Assume that the system satisfies a set of n second-order, time-independent, equations of motion:

$$\ddot{x}_i = R_i(x_1,...,x_n,\dot{x}_1,...,\dot{x}_n), \qquad i=1,...,n, \tag{1}$$

where a single dot above a variable means the time derivative, and a double dot means the second time derivative.

Introducing velocities $(v_1,...,v_n)$ defined by

$$v_i = \dot{x}_i, \qquad i=1,...,n, \tag{2}$$

we can rewrite the equations (1) and (2) as

$$\begin{aligned} \dot{v}_i &= R_i(x_1,...,x_n,v_1,...,v_n) \\ \dot{x}_i &= v_i, \end{aligned} \qquad i=1,...,n. \tag{3}$$

A constant of motion is a function $C(x_1,...,x_n,v_1,...,v_n)$ that satisfies the condition

$$\dot{C} = 0, \tag{4}$$

where the complete time derivative in (5) is calculated as

$$\dot{C} = \sum_{i=1}^{n} \frac{\partial C}{\partial x_i} \cdot \dot{x}_i + \sum_{i=1}^{n} \frac{\partial C}{\partial v_i} \cdot \dot{v}_i = \sum_{i=1}^{n} \frac{\partial C}{\partial x_i} \cdot v_i + \sum_{i=1}^{n} \frac{\partial C}{\partial v_i} \cdot R_i,$$

where R_i are the right sides of equations in (3).

A constant $C(x_1,...,x_n,v_1,...,v_n)$ is called (locally) non-trivial, if at least one of its partial derivatives is different from zero for all points of an open set of points $(x_1,...,x_n,v_1,...,v_n)$ under consideration.

III. THE GENERALIZED POISSON BRACKETS THAT MAKE THE GIVEN CONSTANT OF MOTION A HAMILTONIAN

The so called "flow box" or "straightening out" theorem[4] tells us that locally there exists an invertible change of variables, given by

$$z_k = z_k(x_1,...,x_n,v_1,...,v_n), \quad k = 1,...,2n, \tag{5}$$

such that the equations of motion (3), when given using the variables z_k, $k = 1,...,2n$ become

$$\begin{aligned} \dot{z}_1 &= 1 \\ \dot{z}_k &= 0, \qquad k = 2,...,2n. \end{aligned} \tag{6}$$

Now assume we have a non-trivial constant of motion

$$C = C(x_1,...,x_n,v_1,...,v_n) = C(z_1,...,z_{2n}) . \tag{7}$$

Then the complete time derivative of C using variables z_k and equations (6) is

$$\dot{C} = \sum_{k=1}^{2n} \frac{\partial C}{\partial z_k} \cdot \dot{z}_k = \frac{\partial C}{\partial z_1} \cdot 1 + \sum_{k=2}^{2n} \frac{\partial C}{\partial z_k} \cdot 0 = \frac{\partial C}{\partial z_1}.$$

Because the complete time derivative of any constant of motion is zero in any variables, then in variables z_k we get

$$\frac{\partial C}{\partial z_1} = 0 .$$

Since C is supposed to be non-trivial, at least one of its partial derivatives with respect to z_k, $k = 2,...,2n$ is locally non-zero. Without the loss of generality, we may assume that $\frac{\partial C}{\partial z_2} \neq 0$. This means that the relation between the values of C and z_2 given by (7) can be inverted, giving

$$z_2 = z_2(z_1, z_3, ..., z_{2n}, C). \qquad (8)$$

Using (8) we can replace the coordinate z_2 by the value of the constant of motion C (for convenience we use the same symbol C to denote both the function which is a constant of motion, as well as its value for a given point, hopefully without creating too much confusion). In other words, we introduce new variables, w_i , as

$$\begin{aligned} w_1 &= z_1 \\ w_2 &= C \\ w_k &= z_k , \quad k = 2,...,2n. \end{aligned} \qquad (9)$$

In variables (9) the equations (6), and therefore also the equations (1) are given as

$$\begin{aligned} \dot{w}_1 &= 1 \\ \dot{w}_2 &= 0 \\ \dot{w}_k &= 0, \quad k = 2,..., 2n, \end{aligned} \qquad (10)$$

and our constant of motion C is equal to w_2 .

Using the variables (9) we can define the Generalized Poisson Brackets for functions $f = f(w_1,...,w_{2n})$ and $g = g(w_1,...,w_{2n})$ by:

$$\{f, g\} = \sum_{m=1}^{n} \left(\frac{\partial f}{\partial w_{2m-1}} \cdot \frac{\partial g}{\partial w_{2m}} - \frac{\partial f}{\partial w_{2m}} \cdot \frac{\partial g}{\partial w_{2m-1}} \right). \qquad (11)$$

Since the definition (11) is completely analogous to the usual definition of the Poisson Brackets in the canonical variables, it is trivial to check all the required properties of the Generalized Poisson Brackets.

Then by a direct calculation, using the fact that $C = w_2$, from (9) and (11), it is easy to show that

$$\begin{aligned}&\{w_1, C\} = 1\\ &\{w_2, C\} = 0 \\ &\{w_k, C\} = 0, \quad k = 2, \ldots, 2n.\end{aligned} \tag{12}$$

Using equations (10), for any function $f = f(w_1, \ldots, w_{2n})$ we get

$$\overset{\bullet}{f} = \sum_{k=1}^{2n} \frac{\partial f}{\partial w_k} \cdot \overset{\bullet}{w}_k = \frac{\partial f}{\partial w_1} \cdot 1 + \sum_{k=2}^{2n} \frac{\partial f}{\partial w_k} \cdot 0 = \frac{\partial f}{\partial w_1}.$$

Using equations (11) and the fact that $C = w_2$, we get, for the same function $f = f(w_1, \ldots, w_{2n})$:

$$\{f, C\} = \sum_{m=1}^{n} \left(\frac{\partial f}{\partial w_{2m-1}} \cdot \frac{\partial C}{\partial w_{2m}} - \frac{\partial f}{\partial w_{2m}} \cdot \frac{\partial C}{\partial w_{2m-1}} \right) = \sum_{m=1}^{n} \left(\frac{\partial f}{\partial w_{2m-1}} \cdot \frac{\partial w_2}{\partial w_{2m}} - \frac{\partial f}{\partial w_{2m}} \cdot \frac{\partial w_2}{\partial w_{2m-1}} \right) =$$

$$= \frac{\partial f}{\partial w_1} \cdot \frac{\partial w_2}{\partial w_2} = \frac{\partial f}{\partial w_1} \cdot 1 = \frac{\partial f}{\partial w_1}.$$

Therefore, for any function $f = f(w_1, \ldots, w_{2n})$, we have

$$\overset{\bullet}{f} = \{f, C\}. \tag{13}$$

If, as it would be expected when the Hamiltonian is not obtained from a Lagrangian, we use the equation

$$\overset{\bullet}{f} = \{f, H\}, \tag{14}$$

as the equation defining the Hamiltonian, then equation (13) tells us that the non-trivial constant C is indeed the Hamiltonian for the system of equations of motion (10), provided we use the Generalized Poisson Brackets (11). And because the equation of motion, the functions and the Generalized Poisson Brackets are objects which are covariant with the changes of variables, the equation (13) will still hold when transformed to the original variables $(x_1, \ldots, x_n, v_1, \ldots, v_n)$. Therefore the constant of motion C is still the Hamiltonian (meaning that it satisfies the equation (13)) even if we use the original set of variables $(x_1, \ldots, x_n, v_1, \ldots, v_n)$, and Generalized Poisson Brackets (11), properly transformed to these variables.

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